SVD PSEUDO-INVERSE DECONVOLUTION OF TWO-DIMENSIONAL ARRAYS

M. A. Matuson

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Intercollege Research Programs and Facilities
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NAVY DEPARTMENT

NAVAL SEA SYSTEMS COMMAND

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ABSTRACT

This thesis considers the expected matched filter response to a signal transmitted through a communications channel whose average scattering properties are known in terms of a scattering function. The matched filter is treated as an image which has been blurred by the properties of the interrogating signal. Removing this blurring is called deconvolution and is the problem addressed in this thesis. The problem is formulated to allow efficient application of the Singular Value Decomposition (SVD) as a method of deconvolution. It is shown that this form is the identical operation to the standard deconvolution via spectral division. Additionally, the problem of noise in the image is addressed and the trade-off between resolution and noise is discussed.

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The following notation will be adhered to in this thesis.

$\bar{a}$ or $\bar{A}$ is a vector

a_i is the i^{th} element of $\bar{a}$

$\underline{A}$ is a matrix

$$\text{diag}\,(a_1,a_2,\ldots,a_N) = \begin{bmatrix} a_1 & & & & 0 \\ & a_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \cdot \\ & & & & a_N \end{bmatrix}, \text{ a diagonal matrix}$$

Given two mxn general matrices $\underline{A}$ and $\underline{B}$,

$\underline{A}^T$ is the transpose with dimensions nxm

$\underline{A}^H$ is the Hermetian transpose with dimensions nxm

$r(\underline{A})$ is the rank of the matrix $\underline{A}$

$\underline{A}^{-1}$ is the inverse which exists if $m=n=r(\underline{A})$

$\underline{A}^{+}$ is the pseudo-inverse, also called the Moore-Penrose (MP) generalized inverse

$(\underline{A})ij$ is the element of $\underline{A}$ in the i^{th} row and j^{th} column

$\underline{A}\circ\underline{B}$ denotes the element-by-element multiplication, that is if $\underline{C}=\underline{A}\circ\underline{B}$, then $(\underline{C})ij = (\underline{A})ij\ (\underline{B})ij$

$\underline{A}\otimes\underline{B}$ is the direct or tensor product

$\succ\!\frac{\underline{A}}{\underline{B}}\!\prec$ is the element-by-element division, that is if $\underline{C} = \succ\!\frac{\underline{A}}{\underline{B}}\!\prec$, then $(\underline{C})ij = \frac{(\underline{A})ij}{(\underline{B})ij}$.

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Chapter 1

General Introduction

An important signal processing problem which has been receiving increased attention is that of identifying the transmission characteristics of a noisy, dispersive communications channel. In the context of this and many other treatments, these characteristics are modelled as a scattering process and described by the corresponding scattering function. The function describing the channel characteristics, called the channel transfer function, is typically treated as a linear time-varying filter in a noisy channel. This leads directly to the derivation of the scattering function.

The scattering function can be thought of as a description of the time and frequency spreading characteristics of the channel. In an active scheme, a signal is transmitted through the channel to probe the scattering function. The received signal is processed to extract this information. There is, however, no perfect probe. Any signal will distort the scattering function estimate, each according to its own properties. Analogously, a lens used to image a physical object will distort the image. Removing the distortion is called deconvolution and is the central topic of this thesis.

The problem is complicated by the stochastic nature of the channel. Convolution in this and many other cases corresponds to a multiple band-pass filtering operation on the signal of interest. The frequencies in the signal outside these bands are attenuated often to a point below the average noise content. Deconvolution, being the inverse filter operation, tends to artifically accentuate the noise as it attempts to compensate for the lost frequency content. This effect can be severe enough to hide the useful information, rendering the estimate meaningless. For this reason, many methods of deconvolution have been studied[1,9] attempting to avoid this problem.

The method used in this thesis is that of employing the Singular Value Decompsition (SVD) to form an inverse used to deconvolve. The advantage of this method is that it provides a simple way to recognize the regions of missing spectral components, manifest as singularities, and treat them in forming the inverse so as to reduce the distorting effects of the inverse filter. Since it is not possible to recover those spectral components buried in the noise, information is lost. The accuracy of the deconvolved estimate is therefore partially determined by the signal-to-noise ratio of the return.

Leading up to the deconvolution algorithm, a review of the mathematical basis is needed. Chapter 2 begins with the typical model of the communication channel. The concepts of the signal auto-ambiguity function and the matched filter are presented. Because the algorithm is ultimately implemented on the digital computer, the equations are discretized and the remainder of the thesis utilizes this representation. The balance of the chapter presents the problem in matrix form and introduces the SVD and the formulation of the pseudo-inverse.

As an important stepping stone to aid in the understanding of the more complex two-dimensional problem, Chapter 3 presents the simplified one-dimensional case (range spreading only). It will be shown that by writing the convolution utilizing a circulant matrix, the problem of pseudo-inversion deconvolution via the SVD is identical to deconvolution via spectral division.

Chapter 4 addresses the problem of deconvolution in two dimensions. The circulant matrix convolution takes on a more complex form and although the magnitude of calculations required to form the pseudo-inverse increases, it remains an efficient algorithm. It will be shown that the two-dimensional deconvolution via the SVD also is identical to deconvolution via two-dimensional spectral division. A few graphical examples of the algorithm implemented in two-dimensions are given. For the final example, noise was added, and the algorithm's performance in this case is shown. A short discussion of the estimation error is presented and is shown to be in agreement with the actual results.

Chapter 2

Modelling the Channel

2.1 Introduction

The channel to be considered is typically modelled as a noisy, doubly spread channel characterized by a linear time-varying filter and estimated by a scattering function. If the goal is to identify the channel, then a method of estimating the scattering function is required. In the typical active scheme, a known signal transmitted through the channel will be spread in frequency and time and this in the presence of noise is the received signal. This spreading characterizing the channel is called the scattering function. With knowledge of the transmitted signal, the received signal can be processed to extract an estimate of this function.

Although the concept of the scattering function is important to the discussion, a derivation of the function and its properties is not. There are several references[1,12,13] to provide a thorough treatment of the scattering functions which will only be stated in this thesis.

The equations will be immediately discretized allowing the problem to be cast in matrix form for implementation on the computer. To close out the chapter, the statement of the SVD theorem is presented followed by a discussion of its ability to deal with near-singularities in the matrix that are detrimental to forming a useful pseudo-inverse matrix.

2.2 Fundamentals

Consider an analytic signal x(t). It is transmitted through the channel and is modified according to the scattering function associated with the channel. The received signal, y(t), is commonly processed by the narrow-band correlation receiver,

$$m(\phi,\tau) = \left| \int y(t)\ x^{*}(t-\tau)\ e^{-2\pi j\phi t}dt \right|^{2}, \tag{2-1}$$

which is the cross-correlation between y(t) and a time (τ) and frequency (ϕ) shifted version of x(t). This is called the matched filter and is that filter which maximizes the SNR between the output and the input in the presence of white noise.[11]

The matched filter output contains information about the scattering function, but only to a certain accuracy. Measurements of any kind can only be as accurate as the probe used to make the measurements. For example, a physical object can be measured only to within the accuracy of the ruler. In this case, the probe is the transmitted signal which has a finite resolution in both the time and frequency dimensions. This accuracy is given by the auto-ambiguity function,

$$a(\phi,\tau) = \left| \int x(t)\ x^{*}(t-\tau)\ e^{-2\pi j\phi t}dt \right|^{2}, \tag{2-2}$$

where * denotes complex conjugation and is a shift invariant function.

The overall effect is a smearing of the scattering function due to the limited resolution of this ambiguity function. More accurately, the mean matched filter output formed with the returned signals can be written as the double convolution of the ambiguity and scattering functions[12,13], i.e.,

$$E\ \{m(\hat{\phi},\hat{\tau})\} = a(\phi,\hat{\phi},\tau,\hat{\tau}) ** s(\phi,\tau), \tag{2-3}$$

where $\hat{\phi}$ and $\hat{\tau}$ are hypothesized ϕ and τ values and $s(\phi,\tau)$ is the scattering function. In this manner, the convolution can be thought of as a mapping of the scattering function into the matched filter. By finding the inverse mapping operator, the matched filter can be deconvolved to recover an estimate of the scattering function. A good analogy is that of the photographic process. The scattering function corresponds to the objects to be photographed, and the matched filter average to the photograph. The ambiguity function is analogous to the lens which slightly fuzzes the image. The deconvolution process then corresponds to removing the blur caused by the lens resulting in a clear image.

Any one matched filter output is only a single realization of the scattering function due to the stochastic nature of the channel. An estimate of the scattering function requires an average over many realizations of matched filter outputs. If the scattering function is non-changing over multiple interrogations, this may be a simple time average. For a slowly varying scattering function, a moving average or a tracking algorithm, possibly with controlled forgetting, may be employed. The average is generally performed prior to deconvolution, but may be done following deconvolution depending on the situation. In any case, the averaging problem in independent of the deconvolution problem and will not be addressed. It will consequently be dropped hereafter.

Rewriting (2-3) in an integral form,

$$m(\hat{\phi},\hat{\tau}) = \iint_{-\infty}^{\infty} s(\phi,\tau)a(\phi - \hat{\phi},\tau - \hat{\tau})d\tau d\phi. \qquad (2\text{-}4)$$

It is clear that if the ambiguity function were a delta function at $(\hat{\phi},\hat{\tau}) = (0,0)$, no deconvolution would be necessary. For a signal with a near ideal ambiguity function, deconvolution may not be helpful. There are a great many useful signals, however, with spread ambiguity functions where deconvolution can be important.

2.3 Discrete Representation

Because the processing of these signals is generally done digitally, it is appropriate to develop a discrete representation of the continuous equations. Digitalization is equivalent to dividing up the τ-ϕ plane into a finite number of cells with sufficient resolution to allow accurate reconstruction in the continuous domain. The size of the cells must therefore be chosen in accordance with sampling theory.

A signal used for imaging purposes will necessarily be of finite duration and hence can be represented with a finite number of samples. Let the transmitted signal be of length L samples starting at τ=0 and with spacing $\Delta\tau$. Also, suppose the required resolution along ϕ is $\Delta\phi$, centered about ϕ=0, and the total number of samples in this direction is K. Without loss of generality, let both L and K be odd. The discrete representation of Equation (2-2) is written

$$a(m,n) = \left| \sum_{l=0}^{L-1} x(l)x^*(l-n)e^{-2\pi j(m\Delta\phi)(n\Delta\tau)} \Delta\tau \right|^2 ,$$

$$m = \frac{-(K-1)}{2}, \ldots , \frac{(K-1)}{2}$$

$$n = -(L-1), \ldots , (L-1) \tag{2-5}$$

The ambiguity function is thus (2L-1) samples long in the τ direction centered at τ=0, and K samples long in the ϕ direction centered at ϕ=0.

In general, the size of the scattering function is not known. With knowledge of the channel scattering process, and again the sampling theorem, the dimensions of $s(\phi,\tau)$ can be estimated. Let $s(\phi,\tau)$ be centered at $\tau = \phi = 0$ and be P and Q samples long in the τ and ϕ directions, respectively. Again, let P and Q be odd. The discrete representation of Equation (2-4) is

$$m(m,n) = \sum_{l=-P'}^{P'} \sum_{k=-Q'}^{Q'} s(k,l)a(k-m,n-l)\Delta\tau\Delta\phi,$$

$$m = -\left(\frac{Q+K-1}{2}\right), \ldots, \left(\frac{Q+K-1}{2}\right)$$

$$n = -\left(\frac{P+L'-1}{2}\right), \ldots, \left(\frac{P+L'-1}{2}\right) \qquad (2-6)$$

where

$$P' = \frac{P-1}{2}, \; Q' = \frac{Q-1}{2} \text{ and } L' = 2L-1.$$

These functions and the lengths associated with them will be used extensively later on.

2.4 Singular Value Decomposition (SVD)

Having written the two-dimensional discrete convolution as equation (2-6), the next logical step is to write this equation in matrix form. The double convolution will be expressed as a matrix multiplication; hence, the general matrix form of Equation (2-6) will be given as

$$\underline{M} = \underline{AS}, \qquad (2-7)$$

where $\underline{M}$ and $\underline{S}$ are the matched filter and scattering function matrices and $\underline{A}$, the ambiguity function matrix, is the operation which maps $\underline{S}$ into $\underline{M}$. Deconvolution is thus performed by finding a matrix defining the inverse mapping operation.

A suitable matrix would be the inverse of $\underline{A}$, written $\underline{A}^{-1}$. Multiplying both sides of Equation (2-7) by A^{-1} yields

$$\underline{A}^{-1}\underline{M} = \underline{S} \tag{2-8}$$

where $\underline{A}^{-1}\underline{A} = \underline{I}$, the identity matrix. If $\underline{A}$ is not of full rank, that is if one or more of the eigenvalues of $\underline{A}$ are zero, then $\underline{A}$ is called a singular matrix and $\underline{A}^{-1}$ will not exist. Furthermore, if $\underline{A}$ is not square, again $\underline{A}^{-1}$ will not exist. Instead, the pseudo-inverse (also called the Moore-Penrose (MP) generalized inverse), $\underline{A}^{+}$ must be used.

The formulation of $\underline{A}^{+}$ also helps alleviate another major problem. Deconvolution of a single matched-filter produces only a single realization of the scattering function due to the stochastic nature of the process. If the matrix $\underline{A}$ is ill-conditioned, that is if the ratio of maximum to minimum eigenvalues (the conditioning number) is sufficiently large, the deconvolution process may accentuate unwanted noise resulting in a poor estimate of the scattering function. As an analogy, consider deconvolution via spectral division in a simple linear system. The output is the convolution of the input with the system transfer function. If the transfer function spectrum has a wide dynamic range, the division of the output spectrum by the transfer function spectrum will lead to significant errors in the regions where the transfer function spectrum has small values but the response is finite due to noise.

For these two reasons, the singular value decomposition (SVD) is a useful tool. First, a statement of the SVD theorem.[4,8] Note that [.] indicates the complex case.

Let

$$\underline{A} \in R_r^{mxn};[C_r^{mxn}],$$

that is, a real valued;[complex] matrix with dimensions mxn and rank r. Then there exists orthogonal;[unitary] matrices $\underline{U} \in R^{mxm};[C^{mxm}]$ and $\underline{V} \in R^{nxn};[C^{nxn}]$ such that

$$\underline{A} = \underline{U\Lambda V^T};[\underline{U\Lambda V^H}] \tag{2-9}$$

where

$$\underline{\Lambda} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\underline{S} = \mathrm{diag}(\sigma_1,\ldots,\sigma_r) \text{ with}$$

$$\sigma_1, \geq \ldots \geq \sigma_r > 0.$$

The diag(·) operator defines a matrix whose diagonal elements are the values in the parentheses and whose off-diagonal elements are all zero. This is commonly referred to as a diagonal matrix.

The columns of the matrix $\underline{U}$ are the orthonormal eigenvectors of the matrix $\underline{A}^T\underline{A};[\underline{A}^H\underline{A}]$ and are called the left singular vectors, while the columns of the matrix $\underline{V}$ are the orthonormal eigenvectors of $\underline{AA}^T;[\underline{AA}^H]$ and are called the right singular vectors. The numbers $\sigma_1,\cdots,\sigma_r$ and $\sigma_{r+1} = 0,\cdots,\sigma_n = 0$ are called the singular values and are the positive square roots of the eigenvalues of $\underline{A}^T\underline{A};[\underline{A}^H\underline{A}]$ or $\underline{AA}^T;[\underline{AA}^H]$.

Under certain conditions the SVD of $\underline{A}$ takes on a simpler form making its computation less cumbersome. Such a case is that of a real symetric matrix $\underline{A}$ of dimensions NxN. Because of this symmetry,

$$\underline{A}^T\underline{A} = \underline{AA}^T$$

and so

$$\underline{U} = \underline{V}$$

and is of dimension NxN. The SVD of $\underline{A}$ becomes

$$\underline{A} = \underline{U\Lambda U}^H \tag{2-10}$$

where

$$\underline{\Lambda} = \text{diag}\{\sigma_0, \sigma_1, \cdots, \sigma_{N-1}\}. \tag{2-11}$$

Forming the product $\underline{A}^T\underline{A}$, equation (2-10) yields the result

$$\underline{A}^T\underline{A} = \underline{U\Lambda U}^H\ \underline{U\Lambda U}^H = \underline{U\Lambda}^2\underline{U}^H, \tag{2-12}$$

where, because $\underline{U}$ is a unitary matrix,

$$\underline{U}^H = \underline{U}^{-1} \quad \text{and so} \quad \underline{U}^H\underline{U} = \underline{I},$$

and

$$\underline{\Lambda}^2 = \text{diag}\ \{\sigma_0^2, \sigma_1^2, \ldots, \sigma_{N-1}^2\}. \tag{2-13}$$

Clearly the eigenvalues of $\underline{A}^T\underline{A}$ are the squares of the eigenvalues of $\underline{A}$. Therefore the singular values of $\underline{A}$ are also the eigenvalues of $\underline{A}$. This case is of fundamental importance in the sections to follow.

The pseudo-inverse $\underline{A}^+$ has the property that

$$\underline{A}^+\underline{A} = \underline{VI}^+\underline{V}^H \tag{2-14}$$

where $\underline{I}^+$ is the pseudo-identity matrix. $\underline{I}^+$ is equal to the identity matrix $\underline{I}$ if $\underline{A}$ is of full rank ($\sigma_k > 0$ for all k). If not, $\underline{I}^+$ contains zeroes on its diagonal corresponding to the zero valued singular values on the diagonal of $\underline{\Lambda}$. Applying this condition to Equation (2-9),

$$\underline{A}^{+} = \underline{V}\ \underline{\Lambda}^{-1}\ \underline{U}^{H} \tag{2-15}$$

where

$$\underline{\Lambda}^{-1} = \text{diag}\ \{\sigma_1^{-1},\ \ldots,\ \sigma_r^{-1},\ 0,\ \ldots\ 0\} \tag{2-16}$$

having n-r zeroes.

Define the new scalar function,

$$\sigma_i^{+} = \begin{cases} \sigma_i^{-1} & \text{if } \sigma_i > \varepsilon \\ 0 & \text{if } \sigma_i < \varepsilon \end{cases}, \tag{2-17}$$

where ε is an arbitrary yet carefully chosen threshhold value such that $\varepsilon > 0$. If $\varepsilon = 0$, the pseudo-inverse is formed as given in equations (2-15) and (2-16). If $\varepsilon > 0$, the SVD expansion is truncated and the pseudo-inverse is

$$\underline{A}^{+} = \underline{V\Lambda^{+}}\underline{U}^{H} \tag{2-18}$$

where

$$\underline{\Lambda}^{+} = \text{diag}\ \{\sigma_1^{+},\ \ldots,\ \sigma_N^{+}\}. \tag{2-19}$$

If the conditioning number of $\underline{A}$ $(\sigma_{max}/\sigma_{min})$ is small, ε may be set to zero without affecting the integrity of the deconvolution process. If, however, $\underline{A}$ has a large conditioning number, $\underline{A}$ is said to be ill-conditioned, and taking the inverse of the small singular values will amplify the effect of random perturbations, making the deconvolution a very noisy process. Since truncation of the SVD to reduce the introduction of noise also results in loss of structural information, careful consideration must be given to the choice of ε.

Chapter 3

The One-Dimensional Case

3.1 Introduction

Thus far, nothing has been said about the actual form of the ambiguity matrix $\underline{A}$. Obviously the choice of an ordering for $\underline{A}$ is fundamental to the solution. Cast in the right terms, the SVD expansion can be simplified making the problem of deconvolution less computationally cumbersome. As a prelude to understanding the more complicated two-dimensional case, first consider the problem in one-dimension.

The equations derived earlier can be collapsed by letting $\phi = 0$ $(m = 0)$. In this case, only temporal variations are taken into account. Such a representation is certainly valid for the class of signals with poor resolution in ϕ, such as the short tone pulse.

The chapter begins with a review of the Discrete Fourier Transform (DFT) in one dimension. Next, the circulant matrix is introduced. It will be shown that the circulant arises naturally in writing the convolution in matrix form. Being a well-behaved matrix form, the circulant reduces the complexity of the problem and Sections 3.4 and 3.5 detail the process of forming the pseudo-inverse and employing it to deconvolve. Finally, it is shown in section 3.6 that by writing the convolution in circulant form, deconvolution via the pseudo-inverse method is in fact identical in form and complexity to deconvolution via the standard spectral division method.

3.2 The One-Dimensional Discrete Fourier Transform (DFT)

Consider an N-point sequence of uniformly spaced time samples, $x(0), x(1),\ldots,x(N-1)$. The Discrete Fourier Transform (DFT) will be an

N-point sequence of uniformly spaced coefficients, X(0), X(1),...,X(N-1), and the two are related by the following pair of equations;[10]

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-2\pi j \frac{kn}{N}} \qquad k = 0,1,...,N-1 \tag{3-1}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{2\pi j \frac{kn}{N}} \qquad n = 0,1,...,N-1 \tag{3-2}$$

The first is called the forward DFT, and the second defines the inverse DFT, or IDFT.

If these two sequences are used to define the two column vectors,

$$\overline{x} = [x_o,x_1,...,x_{N-1}]^T \tag{3-3}$$

and

$$\overline{X} = [X_1,X_2,...,X_{N-1}]^T, \tag{3-4}$$

where the sample number is now written as a subscript, the DFT operation can be written as a matrix multiplication,

$$\overline{X} = \underline{F}_N\ \overline{x}. \tag{3-5}$$

The new matrix, $\underline{F}_N$, is called the N-point DFT matrix, where

$$(\underline{F}_N)_{kn} = e^{-2\pi j \frac{kn}{N}}, \quad k,n,=0,1,...,N-1. \tag{3-6}$$

From Equation (3-5) the IDFT can be directly written,

$$\overline{x} = \underline{F}_N^H\ \overline{X} \tag{3-7}$$

which is in agreement with the matrix expansion of Equation (3-2) within a constant, specifically $\frac{1}{N}$. $(\cdot)^H$ is the Hermitian transpose operator. This matrix formulation will be useful in the following sections.

3.3 Circulant Matrices

Circulant matrices, or circulants, are a highly tractable class of matrices. Eigenvalues, eigenvectors and inverses are simply and efficiently found, making the circulant a desirable form in which to cast the problem when it is possible to do so. The circulant matrix, $\underline{C}$ is necessarily a square matrix and has the form

$$\underline{C} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & & \ddots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{bmatrix} . \tag{3-8}$$

Because the NxN matrix $\underline{C}$ can be completely specified by a single vector of length N, the notation

$$\underline{C} = \mathrm{circ}\{c_0, c_1, \ldots, c_{N-1}\} = \mathrm{circ}\ \{\underline{c}^T\} \tag{3-9}$$

is used without loss of information. The circulant is a special type of Toeplitz matrix, and often used to approximate and explain the behavior of the latter.[7]

Of the many special properties of the circulant, the following four are relevant to the discussion to follow.

(i) If $\underline{B}$ and $\underline{C}$ are both circulant matrices, then the product $\underline{BC}$ commutes and is also circulant. That is,

$$\underline{D} = \underline{BC} = \underline{CB} = \text{circ } \{\overline{d}\}.$$

(ii) All circulants have the same set of eigenvectors. Specifically, the m^{th} eigenvector of an N x N circulant,

$$\overline{w_m} = N^{-1/2} \, [w^o, w^m, w^{2m}, \ldots, w^{(N-1)m}]^T, \quad m = 0, 1, \ldots, N-1, \tag{3-10}$$

where

$$w = e^{2\pi j \cdot (1/N)} \tag{3-11}$$

and the $N^{-1/2}$ is a normalizing constant. These eigenvectors are the same as the vectors in the IDFT matrix $\underline{F}_N^H$. In fact if the matrix $\underline{U}$ is constructed using the $\overline{w_m}$'s as the columns, then

$$\underline{U} = \{\overline{w}_o \mid \overline{w}_1 \mid \ldots \mid \overline{w}_{N-1}\} = \underline{F}_N^H$$

within a constant $(N^{-1/2})$.

(iii) Corresponding to the eigenvector $\overline{w_m}$, is the eigenvalue λ_m, where

$$\lambda_m = \sum_{k=0}^{N-1} c_k w^{mk}, \; m=0, 1, \ldots, N-1 \tag{3-12}$$

and w is defined the same as above. Simply stated, the eigenvalues are the DFT coefficients of the vector $\overline{c}^T$ defined in (3-9). If

$$\overline{T} = \{\lambda_o, \lambda_1, \ldots, \lambda_{N-1}\}^T \tag{3-13}$$

then

$$\overline{T} = \underline{F}_N\overline{c} \ . \tag{3-14}$$

(iv) As a result of these properties, the SVD of $\underline{C}$, from equation (2-10) becomes,

$$\underline{C} = \underline{F}_N^H \, \underline{\Lambda F}_N, \tag{3-15}$$

where

$$\underline{\Lambda} = \mathrm{diag}\{\lambda_0, \lambda_1, \ldots, \lambda_{N-1}\} = \mathrm{diag}\{(\underline{F}_N\overline{c})^T\}, \tag{3-16}$$

an NxN diagonal matrix of the DFT coefficients of $\overline{c}$.

There is one final point to make here. If $\underline{C}$ is given as a real matrix, the SVD theorem requires that it be decomposable by real matrices. Equation (3-15) makes no provisions for this. Whether $\underline{C}$ be real or complex, it is still decomposed by the complex matrix $\underline{F}_N$ and its Hermitian transpose. The choice of matrices in the statement of the SVD theorem is, however, not unique so that the decomposition may be performed using complex matrices. It is required, however, that any complex matrix used be tranformable into a real valued matrix by means of a unitary transformation to preserve the space.

It is important here that one particular case be considered, that for a real, symmetric circulant matrix $\underline{A}$. This requires that the first row ($\overline{a}$) of $\underline{A}$ be circular symmetric about the time zero point (the first sample), i.e.,

$$a_i = a_{N-i} \qquad i=1,\ldots,\frac{N}{2}-1 \quad N \text{ even}$$

$$\text{or } i=1,\ldots,\frac{N-1}{2} \quad N \text{ odd}. \tag{3-17}$$

For a time sequence defined in this manner, the DFT coefficients will necessarily all be real and symmetric, hence there will be a redundancy in the singular values, i.e.,

$$\sigma_i = \sigma_{N-i} \qquad i = 1,\ldots,\frac{N}{2}-1 \quad N \text{ even}$$

$$\text{or} \quad i = 1,\ldots,\frac{N-1}{2} \quad N \text{ odd} \qquad (3\text{-}18)$$

In this case a new matrix may be formed from $\underline{F}_N^H$ by taking linear combination of the columns (singular vectors) corresponding to redundant singular values. This is equivalent to defining a unitary transformation (a rotation of the basis vectors).

Let these new basis vectors, $\overline{r}_i$, be defined

$$\overline{r}_i = \frac{1}{\sqrt{2}}\left[\overline{w}_i + \overline{w}_{N-i}\right] \quad i = 1,\ldots,\frac{N}{2}-1;\ N \text{ even} \qquad (3\text{-}19)$$

and

$$\overline{r}_{N-i} = j \cdot \frac{1}{\sqrt{2}}\left[\overline{w}_i - \overline{w}_{N-i}\right], \quad i = 1,\ldots,\frac{N}{2}+1;\ N \text{ even} \qquad (3\text{-}20)$$

where $\overline{w}_i$ is defined in equation (3-10) and $j = \sqrt{-1}$. The cases i=0 and $i=\frac{N}{2}$ are special because the singular values σ_o and $\sigma_{N/2}$ are real and unique. The new basis vectors are now

$$\overline{r}_o = \overline{w}_o$$

$$\overline{r}_i = \sqrt{\frac{2}{N}}\ \{1, \cos(2\pi i/N),\ldots,\cos(2\pi i(N-1)/N)\}^T,\ i=1,\ldots,\frac{N}{2}-1$$

$$\overline{r}_{N/2} = \overline{w}_{N/2}$$

$$\overline{r}_{N-i} = \sqrt{\frac{2}{N}}\ \{0, \sin(2\pi i/N),\ldots, \sin(2\pi i(N-1)/N)\}^T,\ i=1,\ldots,\frac{N}{2}-1.$$

$$(3\text{-}21)$$

where

$$L' = L-1 \tag{3-26}$$

and N is taken to be even. The a_i's are the ambiguity function values found from equation (2-5) with $\phi=0$ (m=0). Because the $\phi=0$ slice of the ambiguity is symmetric about $\tau=0$ (n=0),

$$\overline{a} = \{a_0, a_1, \ldots, a_{L'}, 0, \ldots, 0, a_{L'}, \ldots, a_2,\ a_1\}^T, \tag{3-27}$$

that is because

$$a_i = a_{-i},\ i = 1,\ 2,\ \ldots,\ L' \tag{3-28}$$

The ambiguity function matrix,

$$\underline{A} = \text{circ}\ \{\overline{a}^T\} \tag{3-29}$$

is an NxN matrix and so this $\underline{A}$ used in Equation (2-7) completely discribes the linear convolution of $\underline{A}$ and $\underline{S}$. Using Equations (3-15) and (3-16),

$$\underline{A} = \underline{F}_N^H\ \underline{\Lambda}\ \underline{F}_N, \tag{3-30}$$

where

$$\underline{\Lambda} = \text{diag}\ \{(\underline{F}_N\ \overline{a})^T\}, \tag{3-31}$$

and the pseudo-inverse,

$$\underline{A}^+ = \underline{F}_N^H\ \underline{\Lambda}^+\ \underline{F}_N \tag{3-32}$$

from Equations (2-17), (2-18), and (2-19).

3.5 <u>Deconvolution</u>

Going back to the fundamental Equation (2-7), choosing the ordering for $\underline{A}$ given by equation (3-9) dictates the orderings for $\underline{M}$ and $\underline{S}$ as well. If the scattering function is chosen to be P samples long and a (equation (3-27)) has been padded with minimum necessary P-1 zeroes, then,

$$\underline{S} = \{0,\ldots 0,\ s_{-P'},\ldots\ s_0,\ldots\ s_{P'},\ldots,0,\ldots 0\}^T, \tag{3-33}$$

where

$$P' = \frac{P-1}{2} \qquad (P \text{ odd}) \tag{3-34}$$

and s_i is the sequence of samples of the scattering function padded at both ends with L' zeroes. $\underline{S}$ is therefore a vector of length N. The matched filter $\underline{M}$ is now necessarily defined as

$$\underline{M} = \{m_{-N'},\ldots,m_0,\ldots,m_{N'}\}^T \tag{3-35}$$

where

$$N' = \frac{N-1}{2} \tag{3-36}$$

There is one special note at this point. Equation (3-36) requires that N be odd. Since the SVD in Equation (3-30) benefits greatly in computional efficiency from employing an FFT algorithm, N must be made even, i.e., a power of two. In this case, $\bar{a}$ must be padded with an odd number of zeroes, and zeroes must also be placed in the appropriate positions in $\underline{M}$ and $\underline{S}$.

Deconvolution is now performed by multiplying both sides of equation (2-7) by the pseudo-inverse $\underline{A}^+$ to yield

$$\underline{A}^+\ \underline{M} = \underline{S} \tag{3-37}$$

and thus recovering an estimate of the true scattering function with the blurring effects of the signal ambiguity function reduced. Combining equations (3-32) and (3-37) results in

$$\underline{F}_N^H \, \underline{\Lambda}^+ \, \underline{F}_N \, \underline{M} = \underline{S}. \tag{3-38}$$

Working from right to left on the left side of this equation, $\underline{F}_N \, \underline{M}$ defines the DFT of the vector $\underline{M}$. Each of the spectral components of this DFT sequence is then multiplied by the corresponding spectral component of $\underline{\Lambda}^+$, and the resulting vector transformed by the IDFT matrix. Rewriting,

$$\underline{S} = \mathrm{IDFT}\{\mathrm{DFT}(\overline{a}^T)^+ \circ \mathrm{DFT}(\underline{M})\} \tag{3-39}$$

by virtue of Equations (3-16) and (3-27) where the $(\ )^+$ operator is defined in Equation (2-17). The operation $(\ ) \circ (\ )$ is an element by element multiplication of the two vectors.

3.6 Relation to Spectral Division

As was developed earlier, the discrete representation of the matched filter in one dimension is simply the one-dimensional discrete convolution of the ambiguity function and the scattering function, i.e.

$$m(k) = a(k) * s(k). \tag{3-40}$$

With these sequences padded with zeroes as in (3-27) and (3-33), this convolution of time sequences may be written as an element by element multiplication of spectra via the convolution theorem.[10] The above equation becomes

$$\mathrm{DFT}\{m(k)\} = \mathrm{DFT}\{a(k)\} \circ \mathrm{DFT}\{s(k)\}, \tag{3-41}$$

or more simply

$$\mathrm{DFT}\{s(k)\} = \left\rangle \frac{\mathrm{DFT}\{m(k)\}}{\mathrm{DFT}\{a(k)\}} \right\langle, \tag{3-42}$$

where $\succ\!\!\!-\!\!\!-\!\!\!-\!\!\!\prec$ indicates an element by element division. If the spectrum of a(k) has a wide dynamic range, then division by the small values may cause large errors by amplifying the noise in these areas of the matched filter. To avoid this, a new sequence is defined,

$$\mathrm{DFT}\{a(k)\}^{+}\cdot \;=\; \begin{cases} \mathrm{DFT}\{a(k)\}_i^{-1} & \text{if } >\varepsilon \\ \\ 0 & \text{if } <\varepsilon \end{cases} \qquad (3\text{-}43)$$

where ε is a judiciously chosen threshold value. Replacing this in equation (3-42) and employing an IDFT operator on both sides yields

$$s(k) = \mathrm{IDFT}[\{\mathrm{DFT}\{a(k)\}^{+} \circ \mathrm{DFT}\{m(k)\}] \qquad (3\text{-}44)$$

which is identical to Equation (3-39). Thus it has been shown that the SVD formulation is identical to spectral division and since definitions (3-43) and (2-17) are the same, thresholding serves the same purpose in both methods.

Chapter 4

The Two-Dimensional Case

4.1 Introduction

The more general case of the problem presented in two dimensions must now be discussed. To retain the efficiency of the algorithm in this case, it is appropriate to reformulate the problem using a more complex, yet still amenable matrix form for the matrix $\underline{A}$. With this form and such tools as the tensor product and the two-dimensional DFT, the two-dimensional case is shown to be a highly tractable problem in the context of the SVD.

Section 4.2 introduces the tensor product and the two-dimensional DFT in a useful form. As mentioned earlier, a new matrix form, the block-circulant-with-circulant-blocks form, is presented in section 4.3. The singular value decomposition of this form is then given. Two-dimensional convolution can now be written in two different forms. The first results in a simple circulant matrix $\underline{A}$ and deconvolution reduces to the case presented in Chapter 3. The second employs this new matrix form and sections 4.4-4.6 show that in this case deconvolution via the SVD is identical to deconvolution via two-dimensional spectral division.

The remainder of the chapter presents a few examples implemented on the computer along with a short discussion of the considerations involved in choosing a threshold.

4.2 The Two-Dimensional Discrete Fourier Transform (2DFT)

Before presenting the two-dimensional DFT, a brief review of the tensor or direct product is in order. Consider a $K \times L$ matrix $\underline{A}$ and an

$M \times N$ matrix $\underline{B}$. The direct product, also called the tensor or Kronecker product of $\underline{A}$ and $\underline{B}$, is defined;

$$\underline{A} \times \underline{B} = \begin{bmatrix} a_{11}\underline{B}, & a_{12}\underline{B}, & \ldots, & a_{1L}\underline{B} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{K1}\underline{B}, & a_{K2}\underline{B}, & \ldots, & a_{KL}\underline{B} \end{bmatrix} \tag{4-1}$$

resulting in a matrix with dimensions KM x LN. This form is essential to the following discussion.

Consider an $M \times N$ matrix $\underline{x}$. The two-dimensional Discrete Fourier Transform (2DFT) is a matrix $\underline{X}$ also of size $M \times N$ and is found using the double summation[10],

$$X(k,l) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(m,n) e^{-2\pi j \left[\frac{km}{M} + \frac{ln}{N}\right]} \tag{4-2}$$

where

$$0 \leq k \leq M-1 \text{ and } 0 \leq l \leq N-1. \tag{4-3}$$

It is required that the zero-shift element be in the upper left corner, i.e., x(0,0). If this sum is split and written,

$$X(k,l) = \sum_{n=0}^{N-1} \left[\sum_{m=0}^{M-1} x(m,n) e^{-2\pi j \frac{km}{M}} \right] e^{-2\pi j \frac{ln}{N}}, \tag{4-4}$$

it can easily be seen that the term in the large brackets is the one-dimensional DFT of each column, a total of N DFT's each of an M-point sequence. The outer sum is clearly the one-dimensional N-point DFT of each of the M rows. Thus the two-dimensional DFT can be performed simply by performing a one-dimensional DFT on each column and also on each row. This is the usual method. The number of complex multiplication required to perform the 2DFT via FFT routines is

$$N(M\log_2 M) + M(N\log_2 N),$$

or more simply

$$MN\log_2(MN), \qquad (4\text{-}5)$$

using the usual $N\log_2 N$ rule[10].

It will be very useful to be able to write the 2DFT in matrix form as was done in the one-dimensional case. First a new operation must be defined, the ravel. If $\underline{A}$ is an $M \times N$ matrix, the ravel of $\underline{A}$ is formed by stacking the rows of $\underline{A}$ end to end to form one column vector of length MN. That is,

$$\text{rav}[\underline{A}] = \{a_{00}, a_{01}, \ldots, a_{0(N-1)}, a_{10}, a_{11}, \ldots, a_{(M-1)(N-1)}\}^T = \overline{a} \qquad (4\text{-}6)$$

where the indices begin at zero instead of one and are written as an integer pair of subscripts. The inverse operation is

$$\text{irav}\{a_{00}, a_{01}, \ldots, a_{0(N-1)}, a_{10}, a_{11}, \ldots, a_{(M-1)(N-1)}\}^T$$

$$= \text{ir}\overline{\text{a}}\text{v}\ (a) = \underline{A}. \qquad (4\text{-}7)$$

With this definition, the 2DFT of $\underline{A}$ can be written

$$\overline{b} = \underline{G}\ \text{rav}[\underline{A}], \qquad (4\text{-}8)$$

where $\overline{b}$ is an MN-point column vector and $\underline{G}$ is the MN x MN element 2DFT matrix. By careful inspection of (4-2),

$$\underline{G} = \underline{F_M} \otimes \underline{F_N}, \qquad (4\text{-}9)$$

the direct product of the two one-dimensional DFT matrices introduced in Chapter 3. The resulting vector $\overline{b}$ is the ravel of the 2DFT of $\underline{A}$, that is if

$$\underline{B} = \text{2DFT}(\underline{A}), \tag{4-10}$$

then

$$\underline{B} = \text{irav}(\overline{b}), \tag{4-11}$$

where $\overline{b}$ is given in (4-8).

4.3 More Circulant Matrices

The more complex form of the two-dimensional DFT over the one-dimensional case makes the circulant form presented in Chapter 3 less useful. A more complex matrix is needed to hold all the additional information of the two-dimensional case. Yet it would be desirable to find a new matrix form in which to cast the problem and still retain the amenable nature of the circulant. It is for this reason that a more complex circulant form is investigated. With a certain amount of foresight the block-circulant-with-circulant-blocks form will now be introduced. It will be shown in a later section that this matrix form is indeed useful in writing the two-dimensional problem.

Let $\underline{A_i}$ be an $N \times N$ circulant matrix. Furthermore let

$$\underline{A} = \text{circ}\ \{\underline{A_0}, \underline{A_1}, \ldots, \underline{A_{M-1}}\}. \tag{4-12}$$

The matrix $\underline{A}$ contains M circulant blocks arranged such that the blocks themselves are circulant. The dimensions of $\underline{A}$ are $MN \times MN$, a necessarily square matrix called a block-circulant-with-circulant-blocks matrix of order M, N. More simply, $\underline{A}$ is said to be in $\text{BCCB}_{M,N}$. A short example at this point is highly instructive.

Example 1

Consider the following construction. Let

$$\underline{A}_0 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad \underline{A}_1 = \begin{bmatrix} c & d \\ d & c \end{bmatrix}, \text{ and } \underline{A}_2 = \begin{bmatrix} e & f \\ f & e \end{bmatrix},$$

then

$$\underline{A} = \text{circ}\,\{\underline{A}_0, \underline{A}_1, \underline{A}_2\} = \begin{bmatrix} a & b & c & d & e & f \\ b & a & d & c & f & e \\ e & f & a & b & c & d \\ f & e & b & a & d & c \\ c & d & e & f & a & b \\ d & c & f & e & b & a \end{bmatrix}.$$

$\underline{A}$ is in $BCCB_{3,2}$. Notice that a matrix in BCCB is not necessarily a circulant.

As an extension of the properties of a circulant given in Chapter 3, the following theorem[3] can be derived.

Let $\underline{A}$ be a matrix in $BCCB_{M,N}$ constructed as given by (4-12). Furthermore, let $\underline{\Lambda}_{k+1}$, k=0, ..., M-1 be the diagonal matrix of eigenvalues of block $\underline{A}_k$. The matrix $\underline{A}$ is diagonizable by the unitary matrix $\underline{F}_M \otimes \underline{F}_N$ and the diagonal matrix of eigenvalues of $\underline{A}$ is given by

$$\underline{\Lambda} = \sum_{k=0}^{M-1} \underline{W}_M^k \otimes \underline{\Lambda}_{k+1} \tag{4-13}$$

where

$$\underline{W}_M^k = \text{diag}(1, w^k, w^{2k}, \ldots, w^{(M-1)k}) \tag{4-14a}$$

and

$$w = e^{2\pi j/M} \tag{4-14b}$$

A valid eigenvalue decomposition of $\underline{A}$ is therefore,

$$\underline{A} = (\underline{F}_M \otimes \underline{F}_N)^H \, \underline{\Lambda} \, (\underline{F}_M \otimes \underline{F}_N). \tag{4-15}$$

With this construction in hand it is now appropriate to investigate orderings of the two-dimensional ambiguity function matrix.

4.4 The Pseudo-Inverse in Two Dimensions

There are two distinct orderings of the ambiguity matrix to be considered. The first is a more straightforward method and results in a simple circulant matrix. Although this form permits easy calculation of the singular values, the second form to be investigated is clearly the preferable of the two. This second ordering results in a matrix in BCCB form also permitting easy calculation of the singular values. It will be shown that this form is less computationably taxing and that deconvolution via the SVD in this case is identical to deconvolution via spectral division.

Consider the auto ambiguity function defined by equation (2-5). This real-valued function has an inherent symmetry about a_{oo}, that is,

$$a_{ij} = a_{(-i)(-j)}. \tag{4-16}$$

It will be shown that this fact results in a real, symmetric matrix which necessarily has all real eigenvalues. Another important consideration is that of zero padding. The ambiguity function is given to be K samples long in the ϕ direction (m) and 2L-1 samples long in the τ direction (n). If the scattering function is assumed to be P and Q samples long in the ϕ and τ directions, respectively, then the ambiguity function matrix must be zero padded to be a minimum of M and N samples long in the ϕ and τ directions, where

$$M = K + P - 1 \text{ and } N = (2L - 1) + Q - 1. \qquad (4\text{-}17)$$

Typically, however, M and N will be chosen to be the nearest power of two greater than these minimums to permit use of the FFT routines. With these two considerations, the following vectors are now defined.

$$\begin{array}{ll}
\bar{a}_{-M'} & = (0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, 0)^T \\
\vdots & \qquad\qquad\qquad \vdots \\
\bar{a}_{-K'-1} & = (0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, 0)^T \\
 & \\
\bar{a}_{-K'} & = (0,\ldots,0, a_{K'-L'}, \ldots, a_{K'-1}, a_{K'0}, a_{K'1}, \ldots, a_{K'L'}, 0, \ldots, 0)^T \\
\vdots & \qquad\qquad\qquad \vdots \\
\bar{a}_0 & = (0,\ldots,0, a_{0L'}, \ldots, a_{01}, a_{00}, a_{01}, \ldots, a_{0L'}, 0, \ldots, 0)^T \\
\vdots & \qquad\qquad\qquad \vdots \\
\bar{a}_{K'} & = (0,\ldots,0, a_{K'L'}, \ldots, a_{K'1}, a_{K0}, a_{K'-1}, \ldots, a_{K'-L'}, 0, \ldots, 0)^T \\
 & \\
\bar{a}_{K'+1} & = (0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, 0)^T \\
\vdots & \qquad\qquad\qquad \vdots \\
\bar{a}_{M'} & = (0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, 0)^T
\end{array} \qquad (4\text{-}18)$$

where

$$K' = \frac{K-1}{2}, \; L' = L - 1, \text{ and } M' = \frac{M-1}{2} \qquad (4\text{-}19)$$

Each vector is padded with $N - 2L + 1$ zeroes and so there are M vectors each with length N. Notice K, N and M are chosen to be odd, but they can just as easily be chosen to be even.

Let the vector $\bar{a}$ be defined as the concatenation of all of these vectors, that is, let

$$\bar{a} = \{\bar{a}^T_{-M'}, \ \ldots, \ \bar{a}^T_0, \ \ldots, \ \bar{a}^T_{M'}\}^T. \tag{4-20}$$

Furthermore, let a new vector $\bar{a}'$ be defined as the vector $\bar{a}$ cyclic shifted to make a_{00} the first element. The ambiguity function matrix is then defined,

$$\underline{A} = \mathrm{circ}\{\bar{a}'^T\}. \tag{4-21}$$

Since $\bar{a}'$ is a vector of length MN, the matrix $\underline{A}$ has dimensions MN × MN and is a real, symmetric, circulant matrix.

Example 2

To better understand this construction, take the example of a 3 × 3 ambiguity function convolved with a 3 × 3 scattering function. In this case,

$$K = P = 2L - 1 = Q = 3.$$

Therefore,

$$M = 3 + 3 - 1 = 5 \text{ and } N = 3 + 3 - 1 = 5$$

by virtue of (4-17). Furthermore,

$$\begin{aligned}
\bar{a}_{-2} &= (0, \quad 0, \quad 0, \quad 0, \ 0)^T \\
\bar{a}_{-1} &= (0, \ a_{1-1}, \ a_{10}, \ a_{11}, \ 0)^T \\
\bar{a}_0 &= (0, \ a_{01}, \ a_{00}, \ a_{01}, \ 0)^T \\
\bar{a}_1 &= (0, \ a_{11}, \ a_{10}, \ a_{1-1}, \ 0)^T \\
\bar{a}_2 &= (0, \quad 0, \quad 0, \quad 0, \ 0)^T.
\end{aligned}$$

The matrix $\underline{A}$ is shown in Figure (4-1). It is clearly a 25 × 25 element, real, symmetric, circulant matrix.

M	A																									S
2-2	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	-
2-1	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	-
20	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-
21	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-
22	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	-
1-2	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	-
1-1	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1
10	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	10
11	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	11
12	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-
0-2	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-
0-1	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	-	0-1
00	-	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	-	00
01	-	-	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-	01
02	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-	-
-1-2	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-	-
-1-1	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-	-1-1
-10	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-	-10
-11	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	00	01	-	-	11	10	1-1	-11
-12	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	11	01	-	-	11	10	-
-2-2	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	11	01	-	-	11	-
-2-1	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	11	01	-	-	-
-20	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	11	01	-	-
-21	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	11	01	-
-22	01	-	-	11	10	1-1	-	-	-	-	-	-	-	-	-	-	-	-	1-1	10	11	-	-	01	00	-

$\underline{M} = \underline{A}\ \underline{S}$

Figure 4-1 The expanded form of Equation (4-31) using the first $\underline{A}$ matrix form (4-21). Note that these integer pairs represent array indices. Dashes represent zero-valued elements.

The matrix defined in equation (4-21) can be decomposed as were the circulants from Chapter 3. The singular values are simply the DFT coefficients of the vector $\underline{a}'$. With this construction, however, the DFT coefficients have no physical meaning. In the one-dimensional case, the DFT coefficients described the spectral contents of the ambiguity function, but because in this case the vectors are lined up end to end, the DFT loses this meaning.

There is one more point to mention before moving onto the second construction. Using the standard $N\log_2 N$ rule[10] for describing computational complexity of the FFT, this matrix will requre $MN\log_2 MN$ complex multiplications in computing a pseudo-inverse. This will be used in comparison with the second construction now introduced.

Let the vectors a_i be defined:

$$
\begin{aligned}
\bar{a}_0 &= (a_{00}, a_{01}, \ldots, a_{0L'}, 0, \ldots, 0, a_{0L'}, \ldots, a_{02}, a_{01})^T \\
\bar{a}_1 &= (a_{10}, a_{11}, \ldots, a_{1L'}, 0, \ldots, 0, a_{1-L'}, \ldots, a_{1-2}, a_{1-1})^T \\
&\vdots \\
\bar{a}_{K'} &= (a_{K'0}, a_{K'1}, \ldots, a_{K'L'}, 0, \ldots, 0, a_{K'-L'}, \ldots, a_{K'-2}, a_{K'-1})^T \\
\bar{a}_{K'+1} &= (0, 0, \ldots \ldots \ldots \ldots, 0)^T \\
&\vdots \\
\bar{a}_{K'+1-M} &= (0, 0, \ldots \ldots \ldots \ldots, 0)^T \\
\bar{a}_{-K'} &= (a_{K'0}, a_{K'-1}, \ldots, a_{K'-L'}, 0, \ldots, 0, a_{K'L'}, \ldots, a_{K'2}, a_{K'1})^T \\
&\vdots \\
\bar{a}_{-1} &= (a_{10}, a_{1-1}, \ldots, a_{1-L'}, 0, \ldots, 0, a_{1L'}, \ldots, a_{12}, a_{11})^T
\end{aligned}
\tag{4-22}
$$

where the equation in (4-19) still hold and each vector is padded with a minimum of N-2L+1 zeroes. Also, the indices are written as an integer pair of subscripts. Again, there are M vectors each of length N. Furthermore, let

$$\underline{A}_0 = \text{circ}\{\bar{a}_0^T\}$$

$$\underline{A}_1 = \text{circ}\{\bar{a}_1^T\}$$

$$\vdots$$

$$\underline{A}_{M-1} = \text{circ}\{\bar{a}_{-1}^T\}. \tag{4-23}$$

Finally, the ambiguity function matrix is defined

$$\underline{A} = \text{circ}\{\underline{A}_0, \underline{A}_1, \ldots, \underline{A}_{M-1}\} \tag{4-24}$$

a matrix in $BCCB_{M,N}$.

Example 3

Consider the following example. As in the previous example, let both the ambiguity and scattering function be described by 3 × 3 matrices. Since Equations (4-17) still hold,

$$M = N = 5.$$

The vectors are defined:

$$\begin{aligned}
\bar{a}_0 &= (a_{00}, \ a_{01}, \ 0, \ 0, \ a_{01})^T \\
\bar{a}_1 &= (a_{10}, \ a_{1-1}, \ 0, \ 0, \ a_{11})^T \\
\bar{a}_2 &= (\ 0, \ 0, \ 0, \ 0, \ 0)^T \\
\bar{a}_{-2} &= (\ 0, \ 0, \ 0, \ 0, \ 0)^T \\
\bar{a}_{-1} &= (a_{10}, \ a_{11}, \ 0, \ 0, \ a_{1-1})^T.
\end{aligned}$$

The resulting matrix $\underline{A}$ is shown in Figure (4-2), and is clearly in $BCCB_{5,5}$. Notice that $\underline{A}$ is also real and symmetric guaranteeing real singular values.

With $\underline{A}$ in BCCB form now, Equations (4-13) and (4-15) can be used to decompose the matrix. Let

$$(\underline{\Lambda})_i^+ = \begin{cases} (\Lambda)_i^{-1} & \text{if } (\Lambda)_i > \varepsilon \\ 0 & \text{if } (\Lambda)_i < 0 \end{cases} \qquad (4\text{-}25)$$

where $(\Lambda)_i$ is the i^{th} singular value, that is the i^{th} diagonal element of $\underline{\Lambda}$. The pseudo-inverse of $\underline{A}$, designated $\underline{A}^+$, is calculated as

$$\underline{A}^+ = (\underline{F}_M \otimes \underline{F}_N)^H \, \underline{\Lambda}^+ \, (\underline{F}_M \otimes \underline{F}_N) \qquad (4\text{-}26)$$

where ε is choosen large enough to avoid the harmful effect of ill-conditioning in $\underline{A}$, but small enough to retain sufficient structure for deconvolution.

4.5 Two-Dimensional Deconvolution

With these two orderings for the ambiguity function matrix, the remaining two matrices given in the fundamental equation (2-7) are also determined. Consider the first $\underline{A}$ matrix form. Let

$$\tilde{\underline{A}} = \{\bar{a}_{-M'}, \ldots, \bar{a}_0, \ldots, \bar{a}_{M'}\}^T \qquad (4\text{-}27)$$

where the $\bar{a}_i$ vectors are defined in (4-18). It should be noted that $\tilde{\underline{A}}$ is not the ambiguity function matrix in (2-7). It is the matrix defined in (2-5), the discrete representation of the ambiguity function. The matrix $\underline{A}$ is a ravelled matrix defined in (4-21).

$$
\left[\begin{array}{c}
00\\ 01\\ 02\\ 0{-}2\\ 0{-}1\\ {-}10\\ {-}11\\ {-}12\\ {-}1{-}2\\ {-}1{-}1\\ {-}20\\ {-}21\\ {-}22\\ {-}2{-}2\\ {-}2{-}1\\ 20\\ 21\\ 22\\ 2{-}1\\ 2{-}2\\ 10\\ 11\\ 12\\ 1{-}1\\ 1{-}2
\end{array}\right]
=
\left[\begin{array}{ccccccccccccccccccccccccc}
00 & 01 & - & - & 01 & 10 & 1{-}1 & - & - & 11 & - & - & - & - & - & - & - & - & - & - & 10 & 11 & - & - & 1{-}1\\
01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & -\\
- & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & -\\
- & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11\\
01 & - & - & 01 & 00 & 1{-}1 & - & - & 11 & 10 & - & - & - & - & - & - & - & - & - & - & 11 & - & - & 1{-}1 & 10\\
10 & 11 & - & - & 1{-}1 & 00 & 01 & - & - & 01 & 10 & 1{-}1 & - & - & 11 & - & - & - & - & - & - & - & - & - & -\\
1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & - & -\\
- & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & -\\
- & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & -\\
11 & - & - & 1{-}1 & 10 & 01 & - & - & 01 & 00 & 1{-}1 & - & - & 11 & 10 & - & - & - & - & - & - & - & - & - & -\\
- & - & - & - & - & 10 & 11 & - & - & 1{-}1 & 00 & 01 & - & - & 01 & 10 & 1{-}1 & - & - & 11 & - & - & - & - & -\\
- & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & -\\
- & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & - & -\\
- & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & - & - & - & -\\
- & - & - & - & - & 11 & - & - & 1{-}1 & 10 & 01 & - & - & 01 & 00 & 1{-}1 & - & - & 11 & 10 & - & - & - & - & -\\
- & - & - & - & - & - & - & - & - & - & 10 & 11 & - & - & 1{-}1 & 00 & 01 & - & - & 01 & 10 & 1{-}1 & - & - & 11\\
- & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & - & -\\
- & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1 & -\\
- & - & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & - & 11 & 10 & 1{-}1\\
- & - & - & - & - & - & - & - & - & - & 11 & - & - & 1{-}1 & 10 & 01 & - & - & 01 & 00 & 1{-}1 & - & - & 11 & 10\\
10 & 1{-}1 & - & - & 11 & - & - & - & - & - & - & - & - & - & - & 10 & 11 & - & - & 1{-}1 & 00 & 01 & - & - & 01\\
11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & - & -\\
- & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01 & -\\
- & - & 11 & 10 & 1{-}1 & - & - & - & - & - & - & - & - & - & - & - & - & 1{-}1 & 10 & 11 & - & - & 01 & 00 & 01\\
1{-}1 & - & - & 11 & 10 & - & - & - & - & - & - & - & - & - & - & 11 & - & - & 1{-}1 & 10 & 01 & - & - & 01 & 00
\end{array}\right]
\left[\begin{array}{c}
00\\ 01\\ -\\ -\\ 0{-}1\\ {-}10\\ {-}11\\ -\\ -\\ {-}1{-}1\\ -\\ -\\ -\\ -\\ -\\ -\\ -\\ -\\ -\\ -\\ 10\\ 11\\ -\\ -\\ 1{-}1
\end{array}\right]
$$

$$\underline{M} = \underline{A} \qquad \underline{S}$$

Figure 4-2 The expanded form of Equation (4-31) using the second $\underline{A}$ matrix form (4-24). Note that these integer pairs represent array indices. Dashes represent zero-valued elements.

Let the scattering function have a form identical to $\underline{\tilde{A}}$, that is

$$\underline{\tilde{S}} = \{\bar{s}_{-M'}, \ldots, \bar{s}_0, \ldots, \bar{s}_{M'}\}^T \tag{4-28}$$

where the vectors $\bar{s}_i$ are identical in form to the $\bar{a}_i$'s but with s_{ij} the scattering function values substituted for the a_{ij}'s. Again, $\underline{\tilde{S}}$ is not the scattering function matrix in (2-7). The matched filter matrix, $\underline{\tilde{M}}$, given by (2-6), represents all possible coverings of the matrix $\underline{\tilde{S}}$ by the matrix $\underline{\tilde{A}}$. The $(i, j)^{th}$ element of $\underline{\tilde{M}}$ is the sum of all the products of the coefficients a and s with $\underline{\tilde{A}}$ shifted i elements over and j elements up or down relative to $\underline{\tilde{S}}$.

Example 4

Consider the case given in example two. The matrices $\underline{\tilde{A}}$ and $\underline{\tilde{S}}$ are defined,

$$\underline{\tilde{A}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & a_{1-1} & a_{10} & a_{11} & 0 \\ 0 & a_{0-1} & a_{00} & a_{01} & 0 \\ 0 & a_{-1-1} & a_{-10} & a_{-11} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\underline{\tilde{S}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & s_{1-1} & s_{10} & s_{11} & 0 \\ 0 & s_{0-1} & s_{00} & s_{01} & 0 \\ 0 & s_{-1-1} & s_{-10} & s_{-11} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find $(\underline{\tilde{M}})_{2-1}$, place $\underline{\tilde{A}}$ on $\underline{\tilde{S}}$ and shift $\underline{\tilde{A}}$ one element to the left and 2 elements up,

0	0	0	0	0	
0	a_{1-1}	a_{10}	a_{11}	0	
0	0	0	0	0	0
0	0	a_{-10} s_{1-1}	a_{-11} s_{10}	0	0
0	0	0	0	0	0
	0	s_{-1-1}	s_{-10}	s_{-11}	0
	0	0	0	0	0

,

and sum all the non-zero products.

$$(\underline{\tilde{M}})_{2-1} = a_{10}\ s_{1-1} + a_{1-1}\ s_{10}$$

taking advantage of the symmetry of the ambiguity function as given in (4-16).

The ravel operator defined in equation (4-6) is now employed in the following two definitions. Let

$$\underline{S} = \mathrm{rav}\{\underline{\tilde{S}}\} \tag{4-29}$$

and

$$\underline{M} = \mathrm{rav}\{\underline{\tilde{M}}\}. \tag{4-30}$$

$\underline{M}$ and $\underline{S}$ are actually vectors, but will continue to be written with the underline and treated as matrices. It was with foresight that $\underline{A}$ was constructed as given by (4-21), because now,

$$\underline{M} = \underline{AS}, \tag{4-31}$$

the fundamental equation. The matrix $\underline{A}$ in this case is simply a circulant and the deconvolution,

$$\underline{S} = \underline{A}^{+}\underline{M} \tag{4-32}$$

is identical to the method described in section 3.5. Using the case given in examples two and four along with Equations (4-29) and (4-30), Equation (4-31) is shown in its expanded form in Figure (4-1). There is nothing new here, so the second $\underline{A}$ matrix construction will be discussed.

In a similar manner to the previous case, let

$$\underline{\tilde{A}} = \{\bar{a}_0,\ldots.,\bar{a}_{K'+1},\ldots.,\bar{a}_{-K'},\ldots,\bar{a}_{-1}\}^T \tag{4-33}$$

where the $\bar{a}_i$ vectors are defined in (4-22). Again, let $\underline{\tilde{S}}$ be defined in a similar manner as $\underline{\tilde{A}}$ but with s_{ij} substituted for a_{ij}. The convolution operation producing $\underline{\tilde{M}}$ is different. In the previous case the $(\tilde{M})_{oo}$ element appeared in the center of the matrix. In this case, $(\underline{\tilde{M}})_{oo}$ appears in the upper left corner position in the matrix. This is the result of the ordering chosen for $\underline{A}$, the ravelled ambiguity matrix. It is important that $(\underline{\tilde{M}})_{oo}$, the zero-shift element be in this position when employing the DFT routines as will be done shortly.

$\underline{M}$ and $\underline{S}$ are formed once again as given in (4-29) and (4-30). Continuing the case given in example three, (4-31) is expanded as shown in Figure (4-2).

Using (4-15) and (4-25) the deconvolution in (4-32) can be written;

$$\underline{S} = (\underline{F}_N \otimes \underline{F}_N)^H \, \underline{\Lambda}^{+} \, (\underline{F}_N \otimes \underline{F}_N) \, \underline{M}, \tag{4-34}$$

where the matrix $\underline{A}$ is in $BCCB_{M,N}$. Going back to the definition of the 2DFT, clearly,

$$(\underline{F}_N \otimes \underline{F}_N)\ \underline{M} = (\underline{F}_N \otimes \underline{F}_N)\ \mathrm{rav}\{\tilde{\underline{M}}\} = \mathrm{rav}\{\mathrm{2DFT}(\tilde{\underline{M}})\}. \qquad (4\text{-}35)$$

Since $\underline{\Lambda}^+$ is a diagonal matrix, the product

$$\underline{\Lambda}^+\ [\mathrm{rav}\{\mathrm{2DFT}(\tilde{\underline{M}})\}] \qquad (4\text{-}36)$$

results in a column vector of size N whose i^{th} element is

$$(\underline{\Lambda}^+)_i \cdot [\mathrm{rav}\{\mathrm{2DFT}(\tilde{\underline{M}})\}]_i. \qquad (4\text{-}37)$$

One more term to the left in (4-34) is the I2DFT matrix. The scattering function estimate, $\tilde{\underline{S}}$, can therefore be written

$$\tilde{\underline{S}} = \mathrm{I2DFT}[\mathrm{irav}\{(\underline{\Lambda}^+)_i \cdot [\mathrm{rav}\{\mathrm{2DFT}(\tilde{\underline{M}})\}]_i\}] \qquad (4\text{-}38)$$

which may appear complex, but it will be shown that this equation can be simplified.

4.6 Relation to Two-Dimensional Spectral Division

Deconvolution via spectral division in two dimensions is fundamentally the same as in one dimension. Using the definition in (4-33), (4-29), and (4-30), the convolution is written,

$$\tilde{\underline{M}} = \tilde{\underline{A}} ** \tilde{\underline{S}} \qquad (4\text{-}39)$$

and employing the convolution theorem,

$$\mathrm{2DFT}(\tilde{\underline{M}}) = \mathrm{2DFT}(\tilde{\underline{A}}) \circ \mathrm{2DFT}(\tilde{\underline{S}}) \qquad (4\text{-}40)$$

where $(\cdot)\circ(\cdot)$ is an element by element multiplication. Deconvolution is simply written,

$$\tilde{\underline{S}} = \mathrm{I2DFT}\left[\ \frac{\mathrm{2DFT}(\tilde{\underline{M}})}{\mathrm{2DFT}(\tilde{\underline{A}})}\ \right] \qquad (4\text{-}41)$$

where again >-------< is an element by element division. Let

$$[2DFT(\underline{\tilde{A}})]_i^+ = \begin{cases} [2DFT(\underline{\tilde{A}})]_i^{-1} & \text{if } >\varepsilon \\ 0 & \text{if } <\varepsilon \end{cases} \tag{4-42}$$

and (4-41) becomes;

$$\underline{\tilde{S}} = I2DFT\{[2DFT(\underline{\tilde{A}})]^+ \circ 2DFT(\underline{\tilde{M}})\}. \tag{4-43}$$

Returning to the statement of the SVD for the $BCCB_{M,N}$ matrix embodied in Equations (4-13), (4-14) and (4-15), a careful look at the matrix of singular values, $\underline{\Lambda}$, must be taken. $\underline{\Lambda}_{m+1}$ is the diagonal matrix of singular values of block $\underline{A}_m$. That is,

$$\underline{\Lambda}_{m+1} = \text{diag}\{DFT(\overline{a})_m)\} \tag{4-44}$$

where $\overline{a}_m$ is the first row of the matrix $\underline{A}_m$ defined in (4-23). Rewriting (4-44) using summation notation,

$$(\underline{\Lambda}_{m+1})_l = \sum_{n=0}^{N-1} (\overline{a}_m)_n e^{-2\pi j\left(\frac{nl}{N}\right)} \quad l=0,\ldots,N-1, \tag{4-45}$$

for the l^{th} element of $\underline{\Lambda}_{m+1}$. Looking at one particular element of $\underline{\Lambda}$, specifically $(\underline{\Lambda})_{kl}$, the l^{th} element of block k.

$$(\underline{\Lambda})_{kl} = \sum_{M=0}^{M-1} \underline{W}_m^{km} (\underline{\Lambda}_{m+1})_l$$

$$= \sum_{m=0}^{M-1} \left[\sum_{n=0}^{N-1} (\overline{a}_m)_n e^{-2\pi j\left(\frac{nl}{N}\right)}\right] e^{-2\pi j\frac{km}{M}}$$

and therefore,

$$(\underline{\Lambda})_{kl} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (\bar{a}_m)_n e^{-2\pi j\left[\frac{km}{M} + \frac{nl}{N}\right]}$$

$$l=0,\ldots,N-1 \qquad k=0,\ldots,M-1, \tag{4-46}$$

which is exactly the form of the 2DFT.

The matrix of singular values is now written,

$$\underline{\Lambda} = \text{diag}\{\text{rav}[\text{2DFT}(\tilde{\underline{A}})]\}. \tag{4-47}$$

Putting this result into (4-38) results in,

$$\tilde{\underline{S}} = \text{I2DFT}[\text{irav}\{\text{diag}[\text{rav}\{\text{2DFT}(\tilde{\underline{A}})\}]\}_i \cdot \text{rav}\{\text{2DFT}(\tilde{\underline{M}})\}_i]. \tag{4-48}$$

But since,

$$\text{diag}\ \{\text{rav}(\underline{A})\}_i \cdot \{\text{rav}(\underline{B})\}_i = \text{rav}(\underline{A}) \circ \text{rav}(\underline{B})$$

and

$$\text{irav}\ \{\text{rav}(\underline{A}) \circ \text{rav}(\underline{B})\} = \underline{A} \circ \underline{B},$$

(4-48) is simplifed,

$$\tilde{\underline{S}} = \text{I2DFT}\{[\text{2DFT}(\tilde{\underline{A}})]^+ \circ \text{2DFT}(\tilde{\underline{M}})\} \tag{4-49}$$

precisly the form in (4-43) resulting from deconvolution via spectral division. In back cases, the [$]^+$ operator is defined in (4-42).

In comparison to the first $\underline{A}$ matrix form, (4-21), the number of complex multiplication required to form the pseudo-inverse in this case is

$$K\log_2 N + N\log_2 M. \tag{4-50}$$

This is the result of the fact that there are M-K rows of all zeroes as is clear from looking at figure (4-2). It is not necessary to DFT these rows and for large matrices, this savings can be significant. Form 2 also has the advantage that the pseudo-inverse matrix coefficients retain their physical meaning, i.e., they are the spectral components of the original matrix. This is useful in choosing a threshold, ε, at which to truncate the SVD.

4.7 Examples of the Algorithm

The algorithm is applied in the following manner. A signal x(k) is defined and the ambiguity function a(m,n) is calculated via (2-5). The matched filter is generally calculated as the correlation of the return signal with a time and frequency shifted version of x(k), the continuous case being given in (2-1). It was stated in Chapter 2 that a single return can only produce one realization of the scattering function and so the expectation operator $E\{\cdot\}$ was introduced. Averaging over many interrogations of the channel reduces the variance of the process; hence, the deconvolution algorithm can produce a better estimate of the scattering function. The ideal case is the double convolution of $a(\phi,\tau)$ and $s(\phi,\tau)$ as given in (2-7) and in the examples to follow, this will be used as the matched filter to be deconvolved.

The matrix $\underline{A}$ is formulated using the definition given in (4-24), the second form. For all the cases given, a(m,n) was choosen to be 64 × 1024 samples long in the m and n directions respectively. This includes the required zero padding. With this construction, the algorithm is now embodied in Equations (4-49) and (4-42). It was mentioned earlier that careful consideration must be given to choosing ε. This will be shown graphically.

Three distinct signals were chosen and are listed in Table (4-1). Signal 1 is a VFM consisting of an upchirp linear FM (LFM) of bandwidth 500Hz followed by a downchirp LFM of the same width both centered at 50kHz. Signal 2 is simply a single upchirp LFM with a 500Hz bandwidth. Signal 3 is a bit more complex. It is generated from a Costas Array developed via a Welsh construction[5], and is a six-element signal based on the 3^n mod 7 construction given in the table and will be called a CAW. The result is an ambiguity function with a high resolution main spike surrounded by a pedestal. Surrounding this pedestal is a clear area containing no ambiguity volume. Minor lobes appear outside this clear area region and since they contain relatively little volume, only the main lobe area will be considered.

Three different scattering function were chosen each consisting of a particular number of point scatterers. They are given in Table (4-2). The first example uses signal type 1 (ST1) and scattering function 1 (SF1). Figure (4-3a) shows the matched filter output. 0dB is defined at the peak spectral component of the ambiguity function and the threshold, ε, is defined with respect to this peak. The remainder of Figure (4-3) shows the effect of employing particular thresholds. At -10dB, there is insufficient information retained to deconvolve with any accuracy. In this case deconvolution is not helpful. As the threshold is reduced, the algorithm is able to clean up the matched filter image. Since there is no noise in this case, ε may be choosen as low as desired without any harmful side effects.

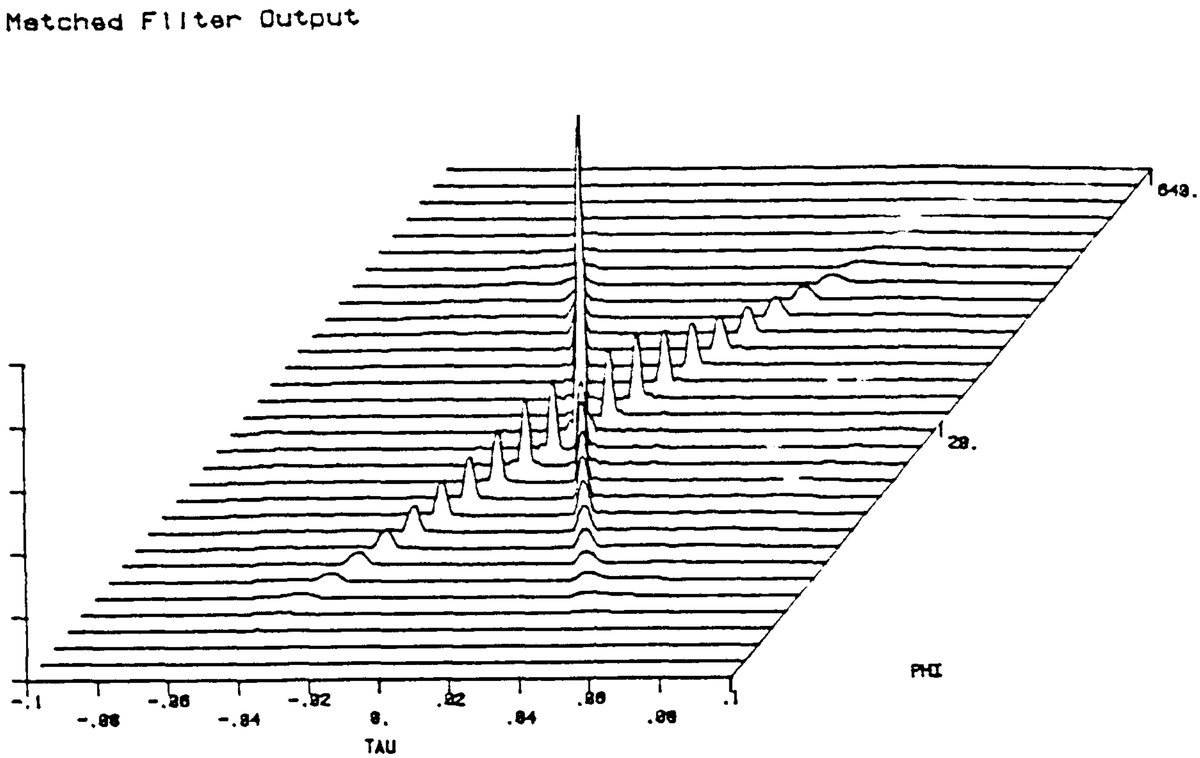
The same is shown to happen for the cases of (ST2 and SF2) and (ST3 and SF3) resulting in Figures (4-4) and (4-5), respectively.

Table 4-1 The signal definitions used in the examples are given here.

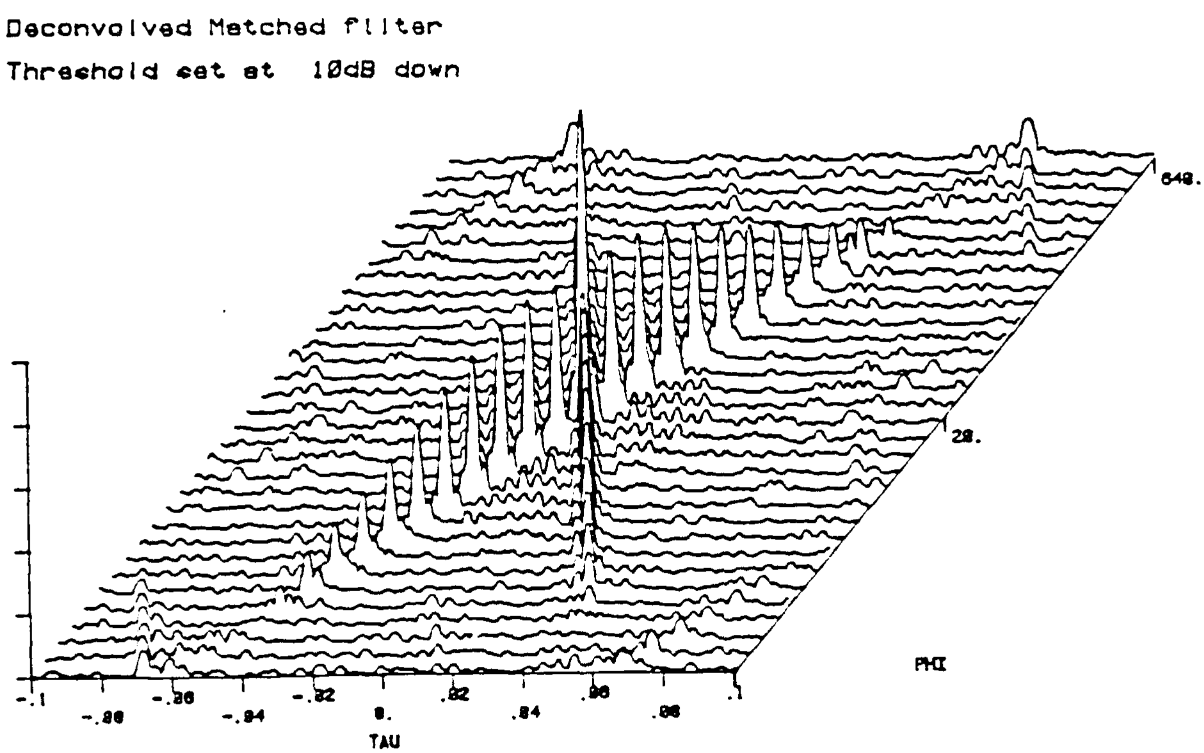
No.	Signal Type	Number of Subpulses	Subpulse Number	Duration in msec.	Band Center in Hz	Bandwidth in Hz
1	VFM	2	1 2	50 50	50000 50000	+500 -500
2	LFM	1	1	50	50000	+500
3	CAW	6	1 2 3 4 5 6	10 10 10 10 10 10	49700 49100 51500 50300 50900 48500	0 0 0 0 0 0

Table 4-2 The scattering functions used in the examples are defined as a collection of point scatterers as given here.

No.	Number of Highlights	Highlight Number	Tau Position in msec.	Phi position in Hz	Relative Strength
1	1	1	0	0	1
2	2	1 2	0 10	0 0	1 1
3	3	1 2 3	0 10 20	0 8 16	1 1 1



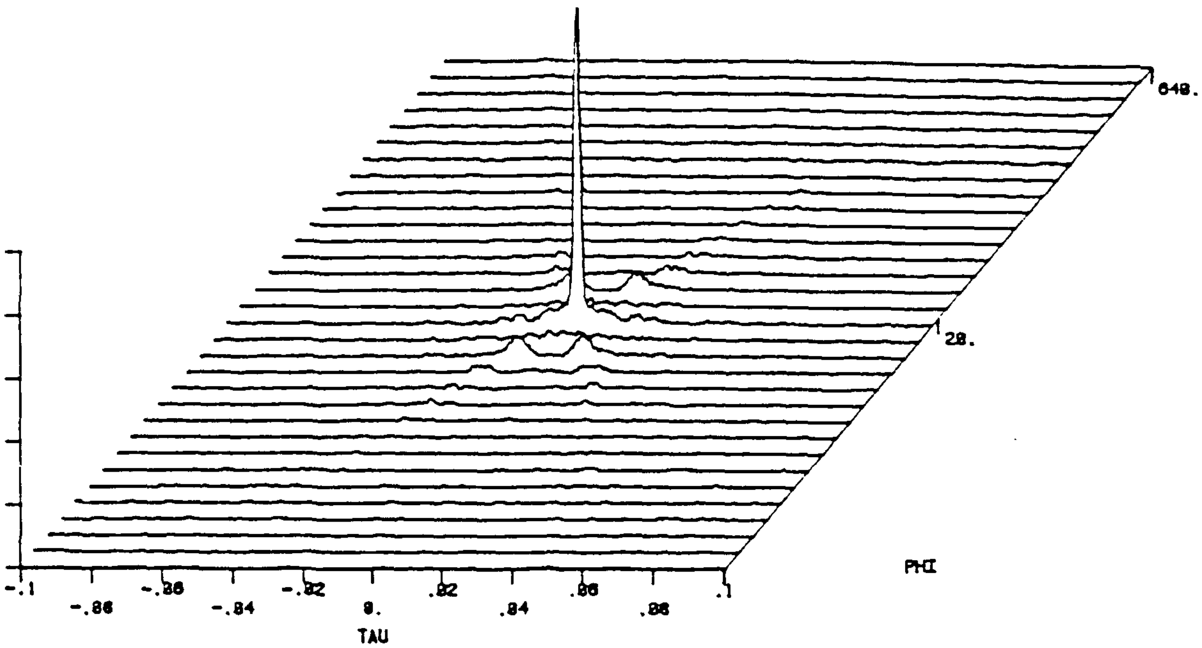
(a)



(b)

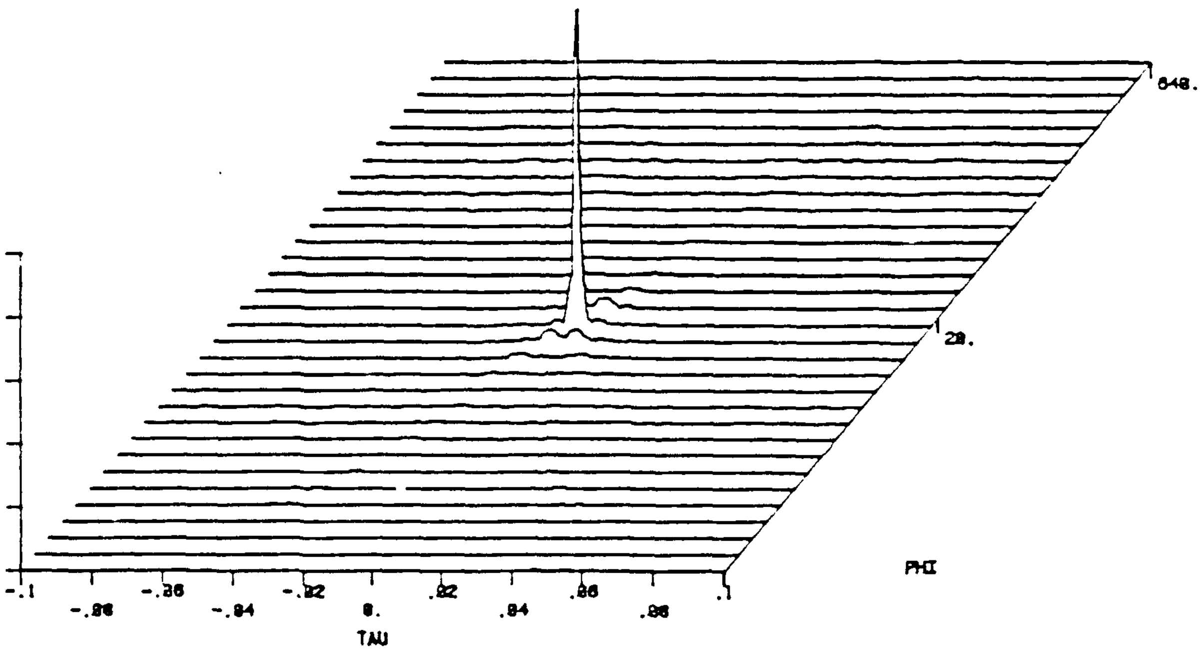
Figure 4-3 Example 1: The VFM and a one highlight scattering function.

Deconvolved Matched filter
Threshold set at 20dB down



(c)

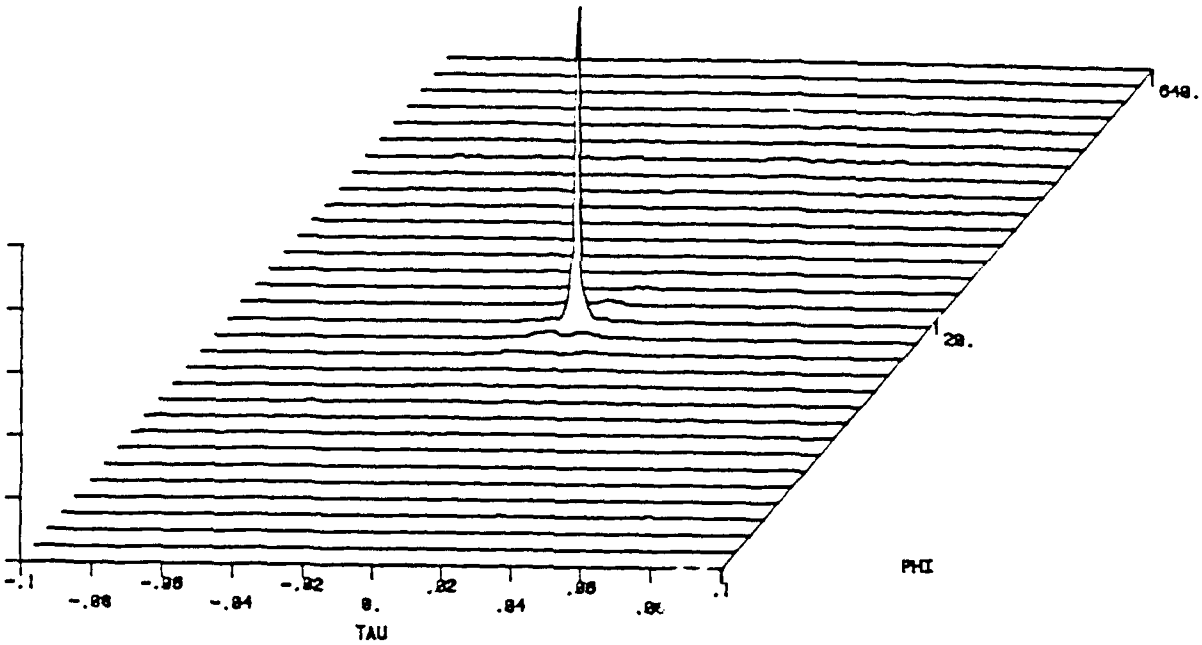
Deconvolved Matched filter
Threshold set at 30dB down



(d)

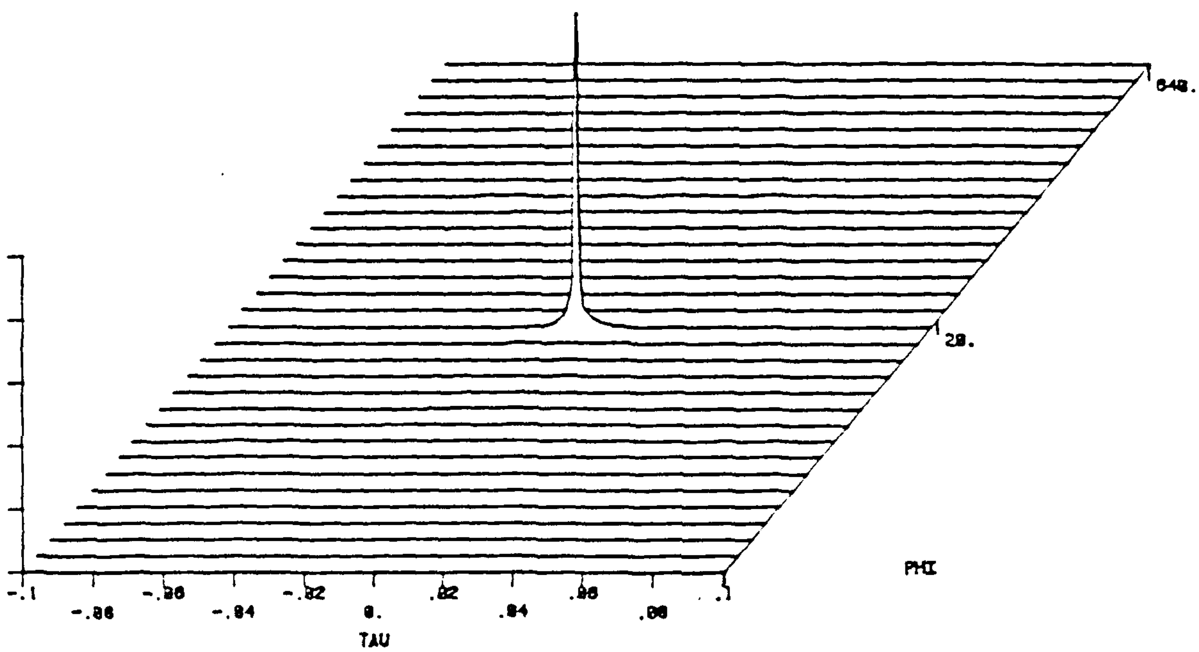
Figure 4-3 Continued

Deconvolved Matched filter
Threshold set at 40dB down



(e)

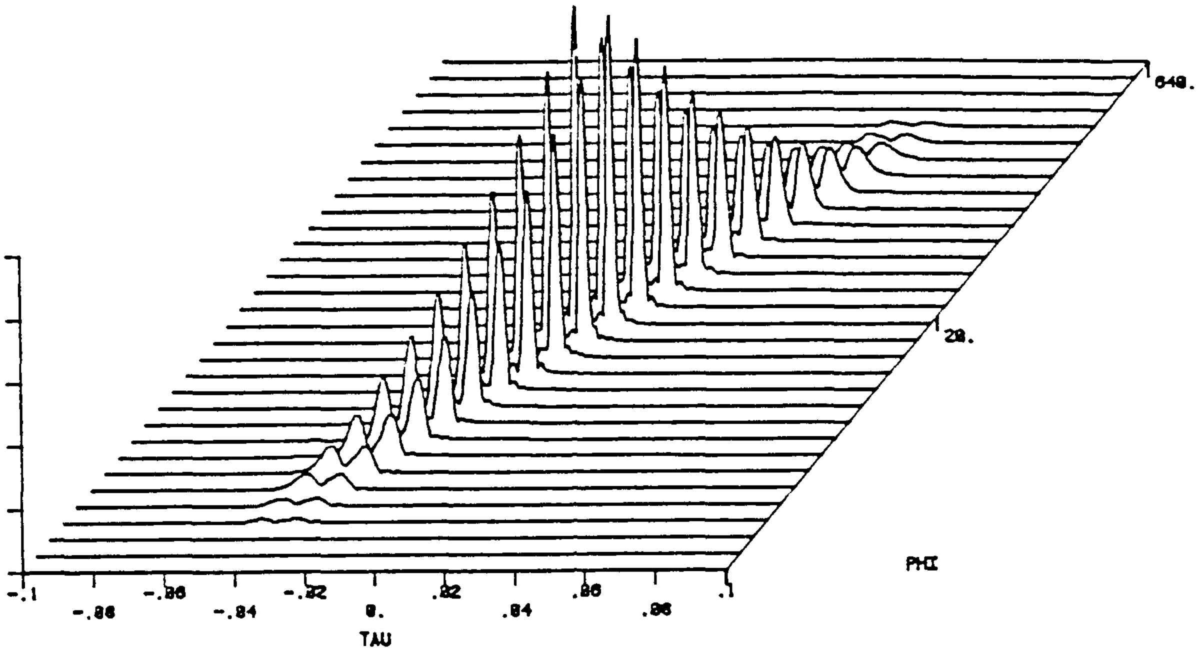
Deconvolved Matched filter
Threshold set at 50dB down



(f)

Figure 4-3 Continued

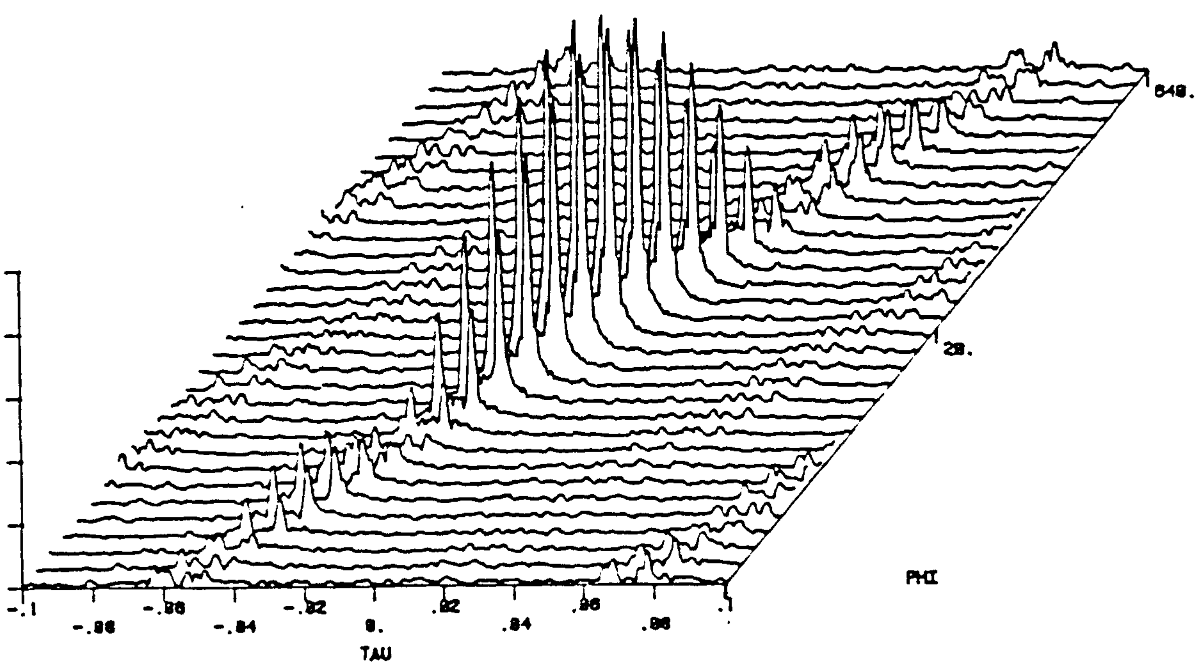
Matched Filter Output



(a)

Deconvolved Matched filter

Threshold set at 10dB down

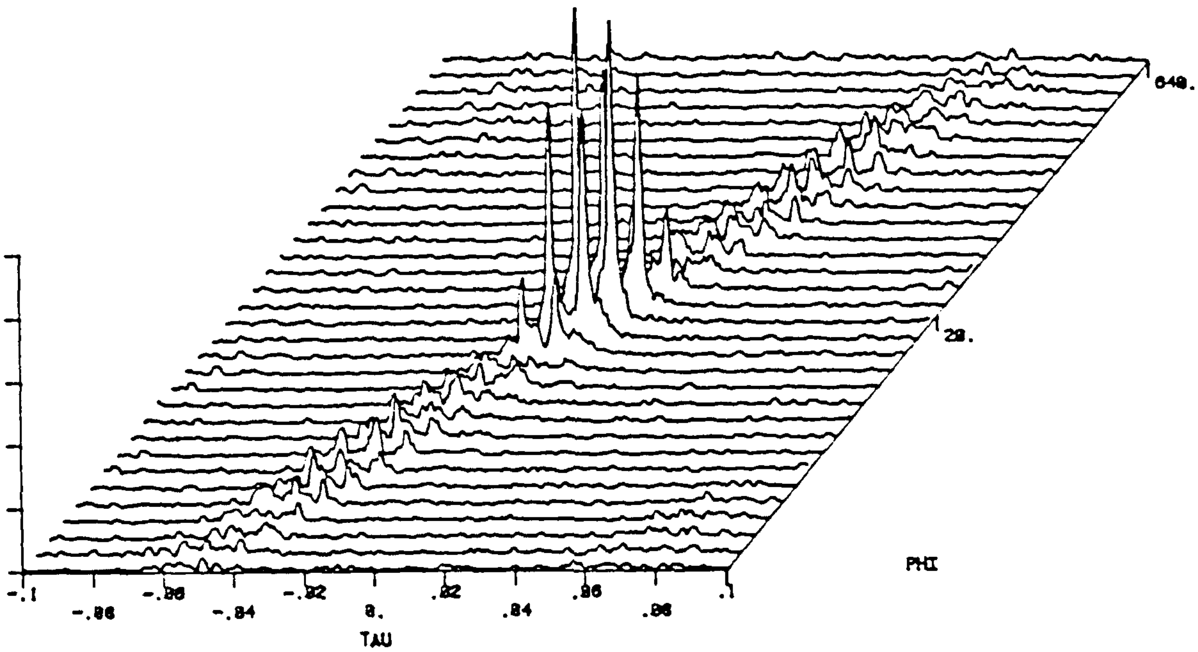


(b)

Figure 4-4 Example 2: The LFM and a two highlight scattering function.

Deconvolved Matched filter

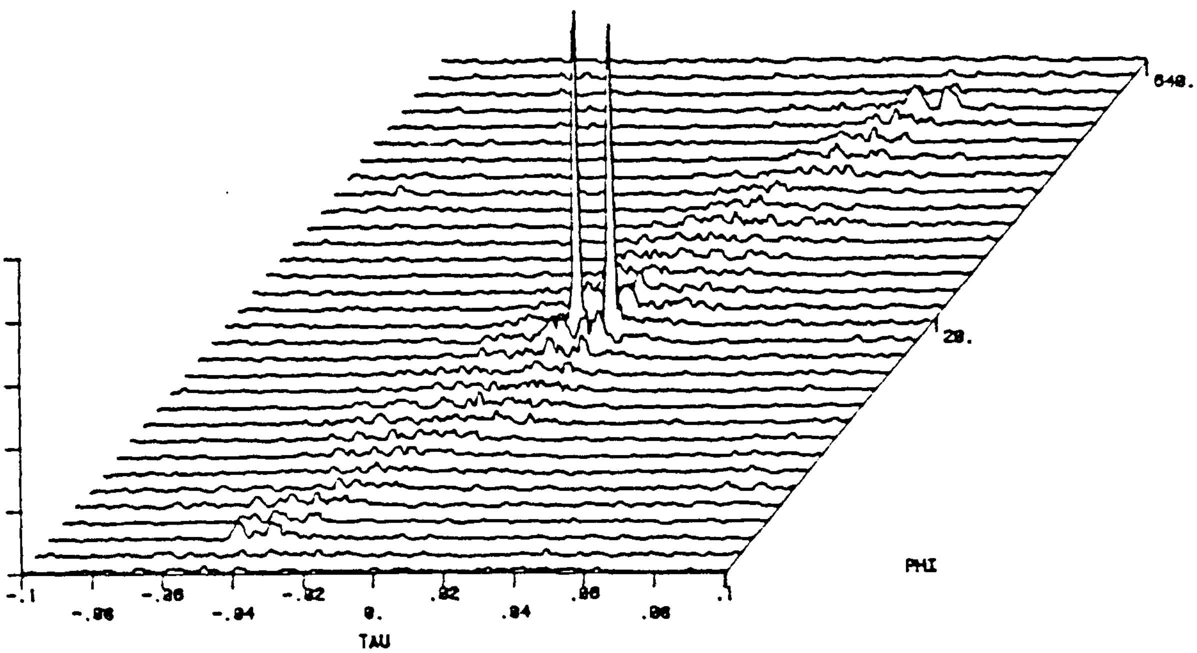
Threshold set at 20dB down



(c)

Deconvolved Matched filter

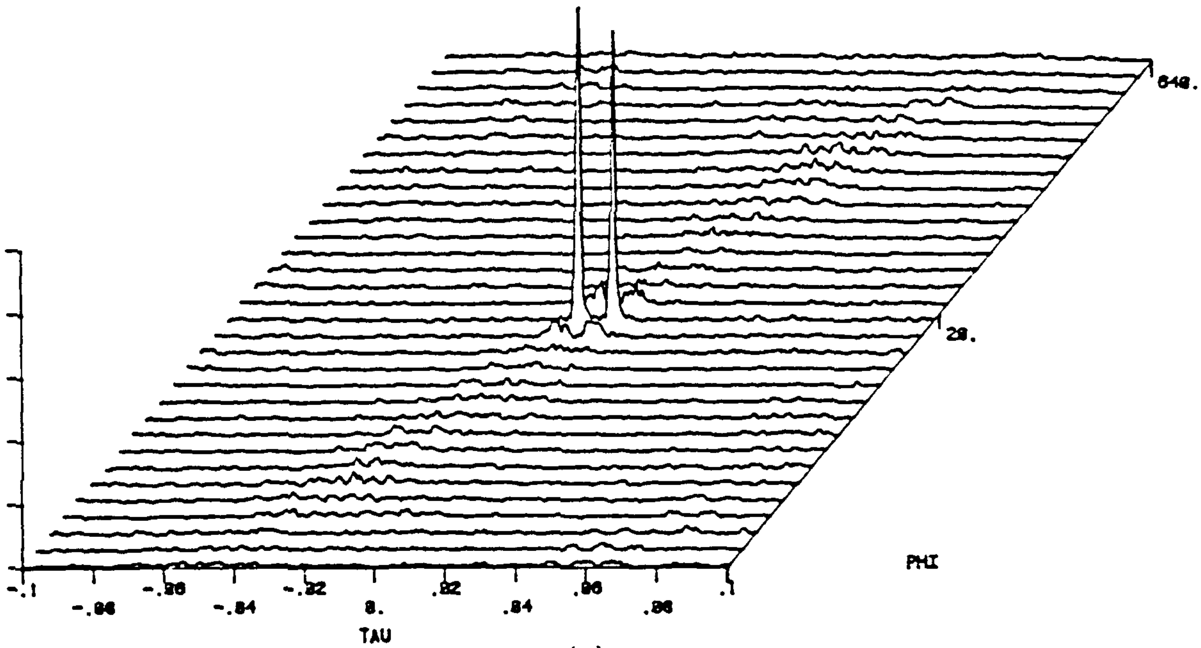
Threshold set at 30dB down



(d)

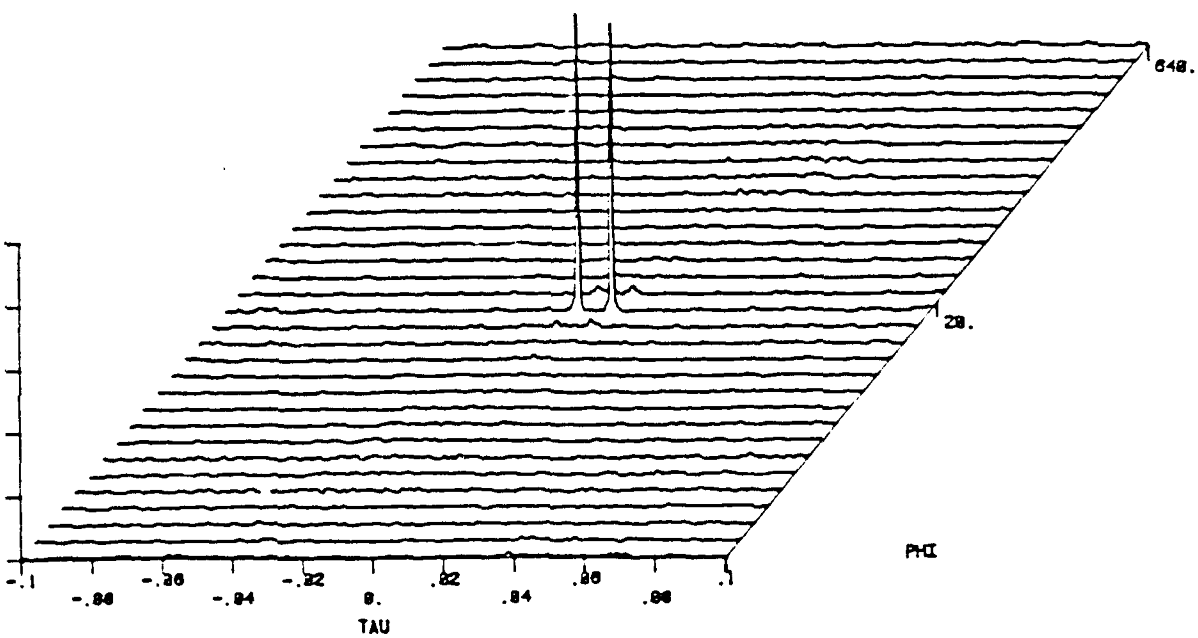
Figure 4-4 Continued

Deconvolved Matched filter
Threshold set at 40dB down



(e)

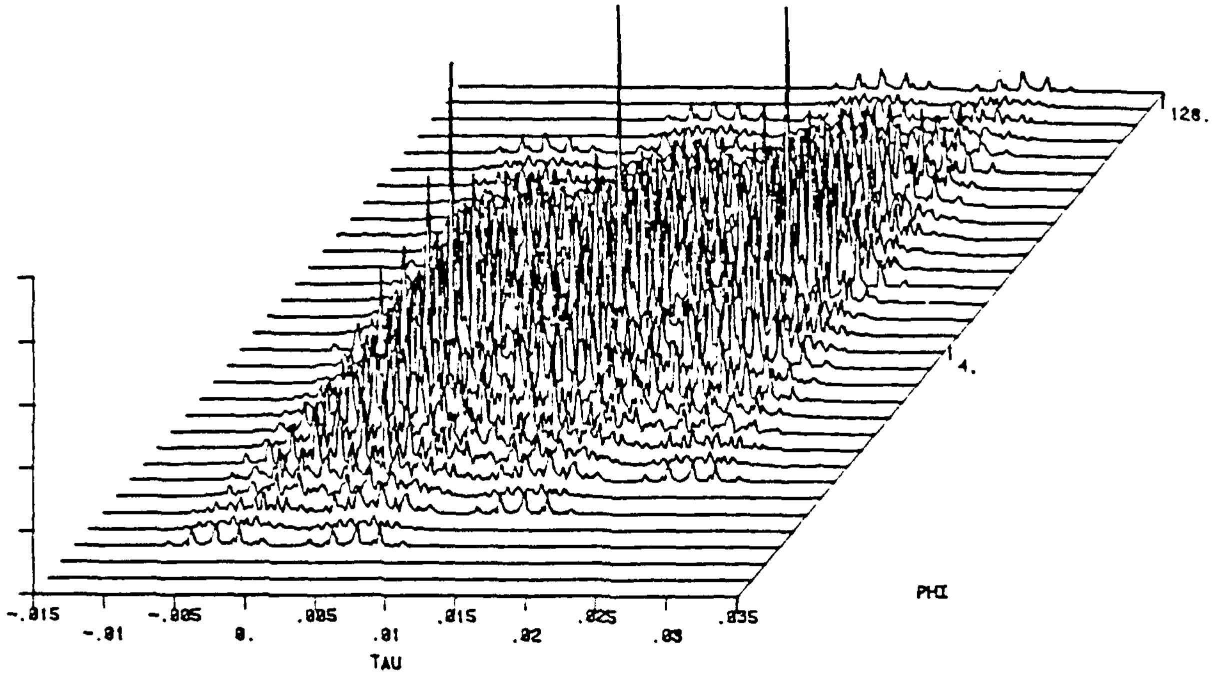
Deconvolved Matched filter
Threshold set at 50dB down



(f)

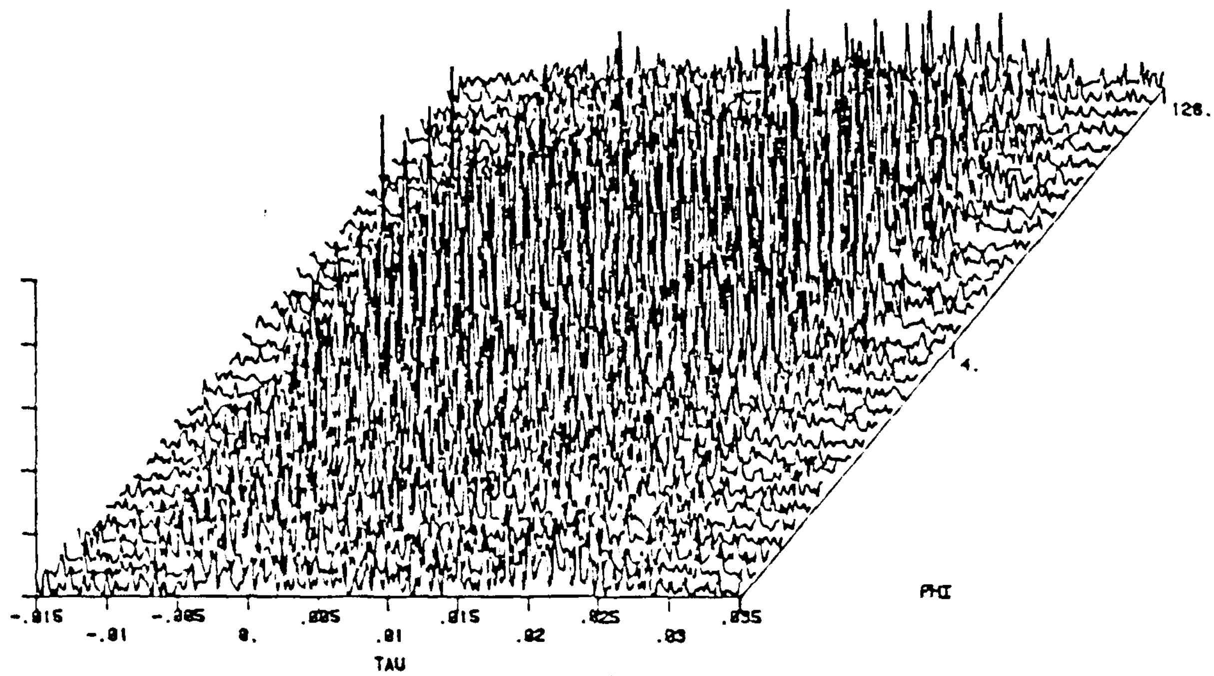
Figure 4-4 Continued

Matched Filter Output



(a)

Deconvolved Matched filter
Threshold set at 10dB down

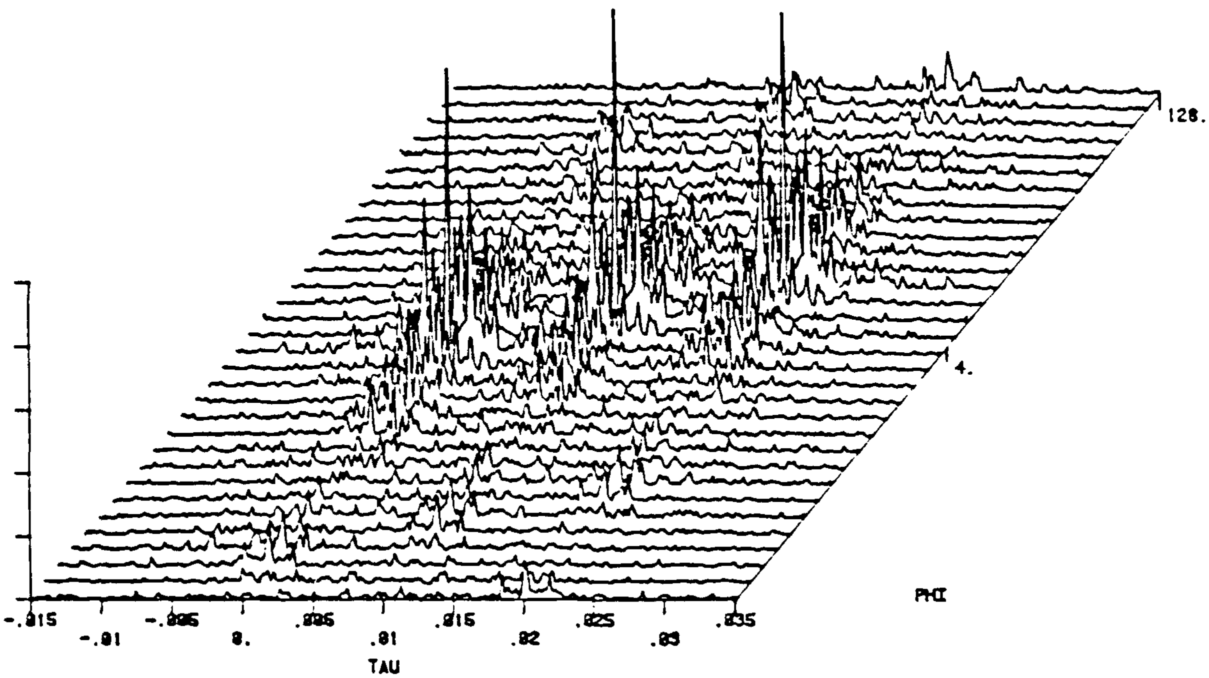


(b)

Figure 4-5 Example 3: The CAW and a three highlight scattering function.

Deconvolved Matched filter

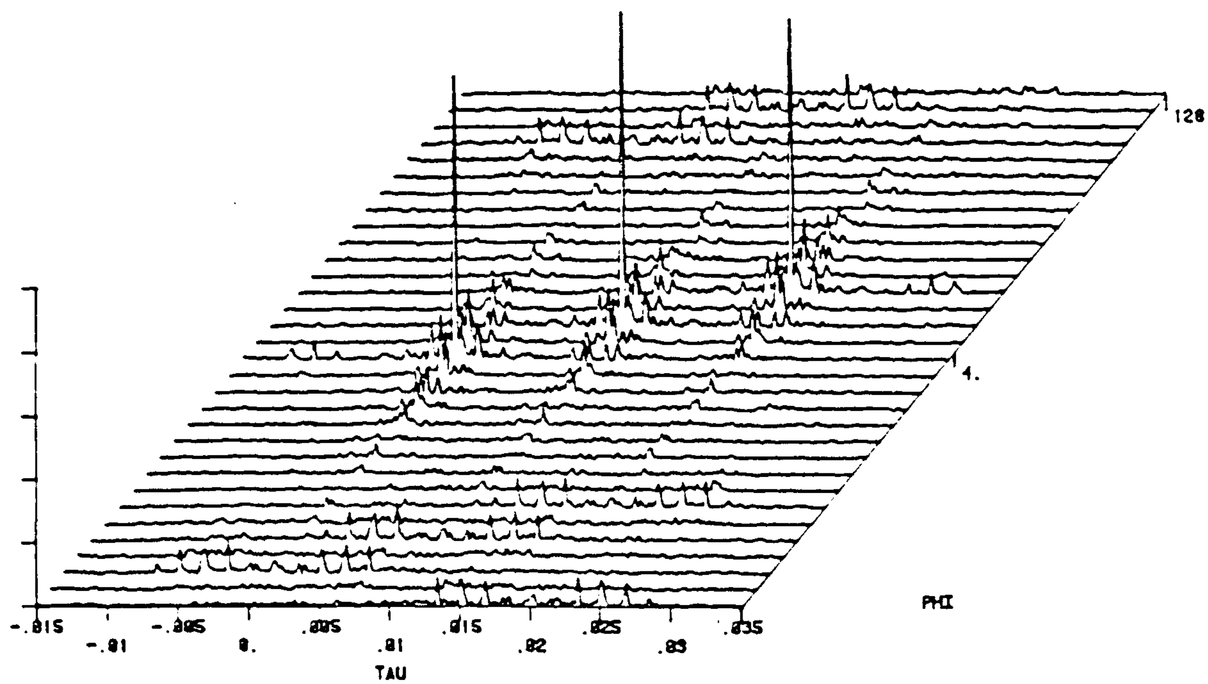
Threshold set at 20dB down



(c)

Deconvolved Matched filter

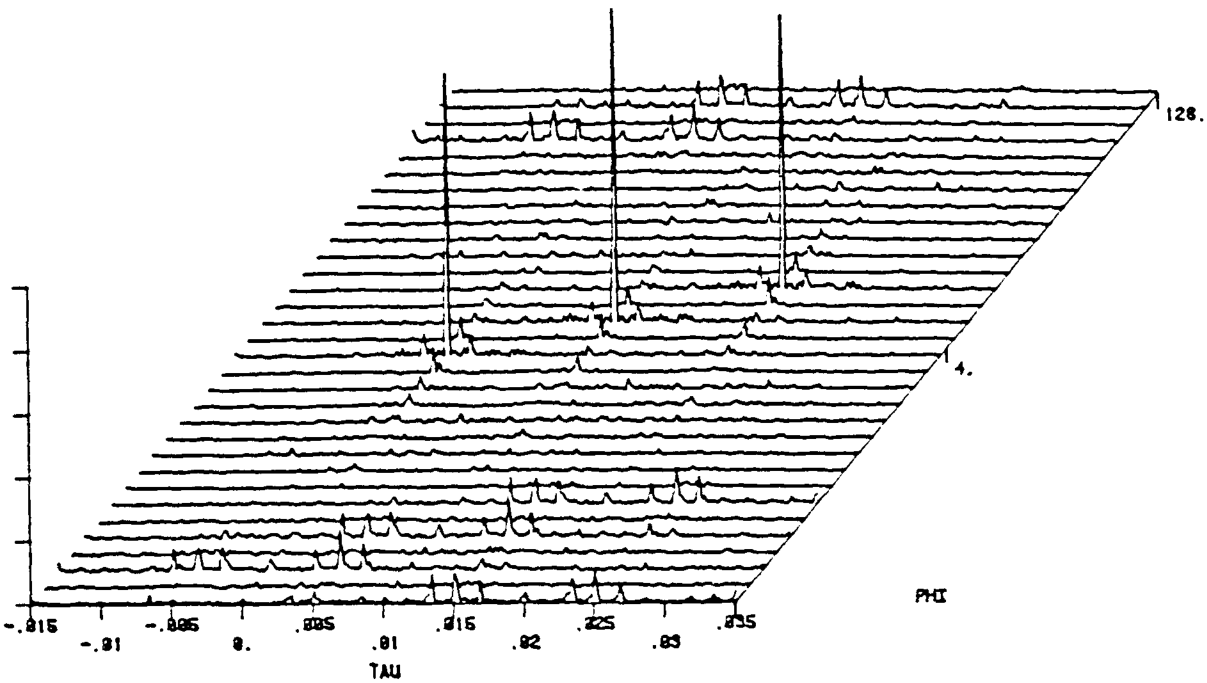
Threshold set at 30dB down



(d)

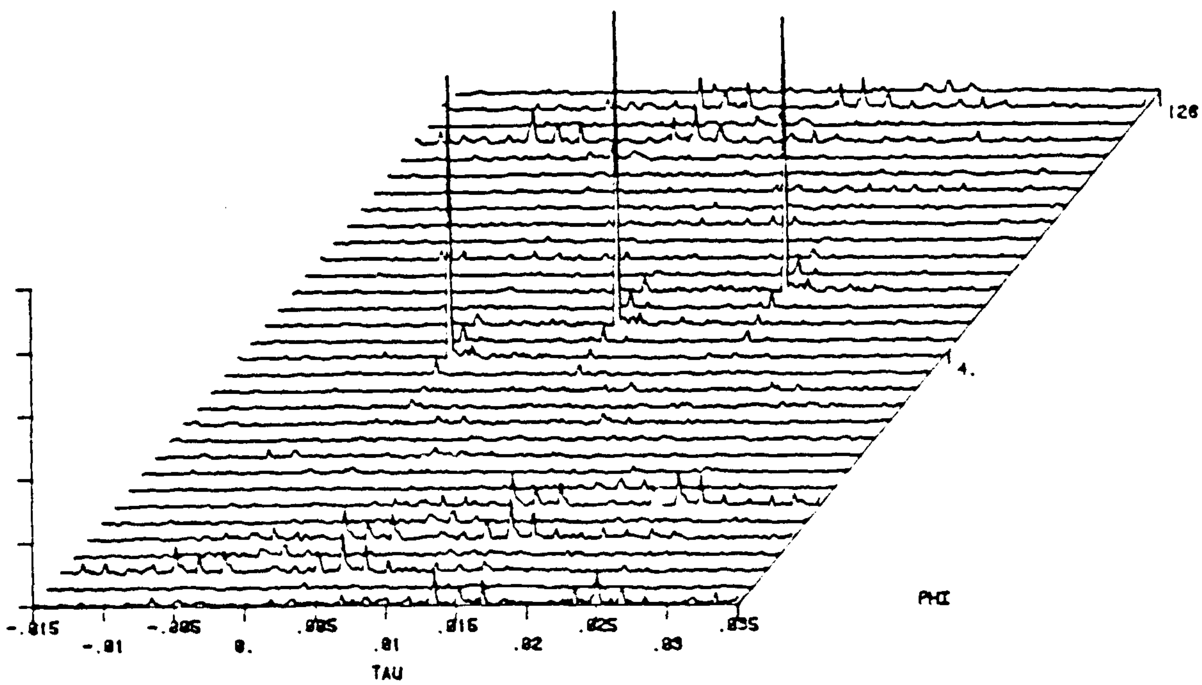
Figure 4-5 Continued

Deconvolved Matched filter
Threshold set at 40dB down



(e)

Deconvolved Matched filter
Threshold set at 50dB down



(f)

Figure 4-5 Continued

Next, noise was added. If it is assumed that Gaussion zero-mean white noise is present at the input to the matched filter, the output noise distribution is chi-square of order two, more commonly called the exponential distribution. It is easy to show this to be the case. The noise in the return is

$$n = X_r + jX_i, \tag{4-51}$$

a complex random variable ($j = \sqrt{-1}$) where both X_r and X_i are zero-mean Gaussian random variables. At the output of the matched filter, the noise

$$z = |n|^2 = X_r^2 + X_i^2 , \tag{4-52}$$

which, as advertised, results in a second order chi-square distribution. The probability density function, $f_z(z)$, is written

$$f_z(z) = \frac{1}{2\sigma^2} \exp\left[\frac{-z}{2\sigma^2}\right]. \tag{4-53}$$

It is easy to show that the mean of this distribution,

$$\mu = \frac{1}{2\sigma^2} \tag{4-54}$$

With this determined, the SNR was choosen to be the peak to average noise in the matched filter, that is

$$SNR = 10 \log \left(\frac{(peak)_{MF}}{\mu}\right). \tag{4-55}$$

If a particular SNR is to be generated, then μ is chosen to be

$$\mu = \frac{(peak)_{MF}}{10^{\frac{SNR}{10}}}, \tag{4-56}$$

and the distribution in (4-53) used to generate noise into the matched filter.

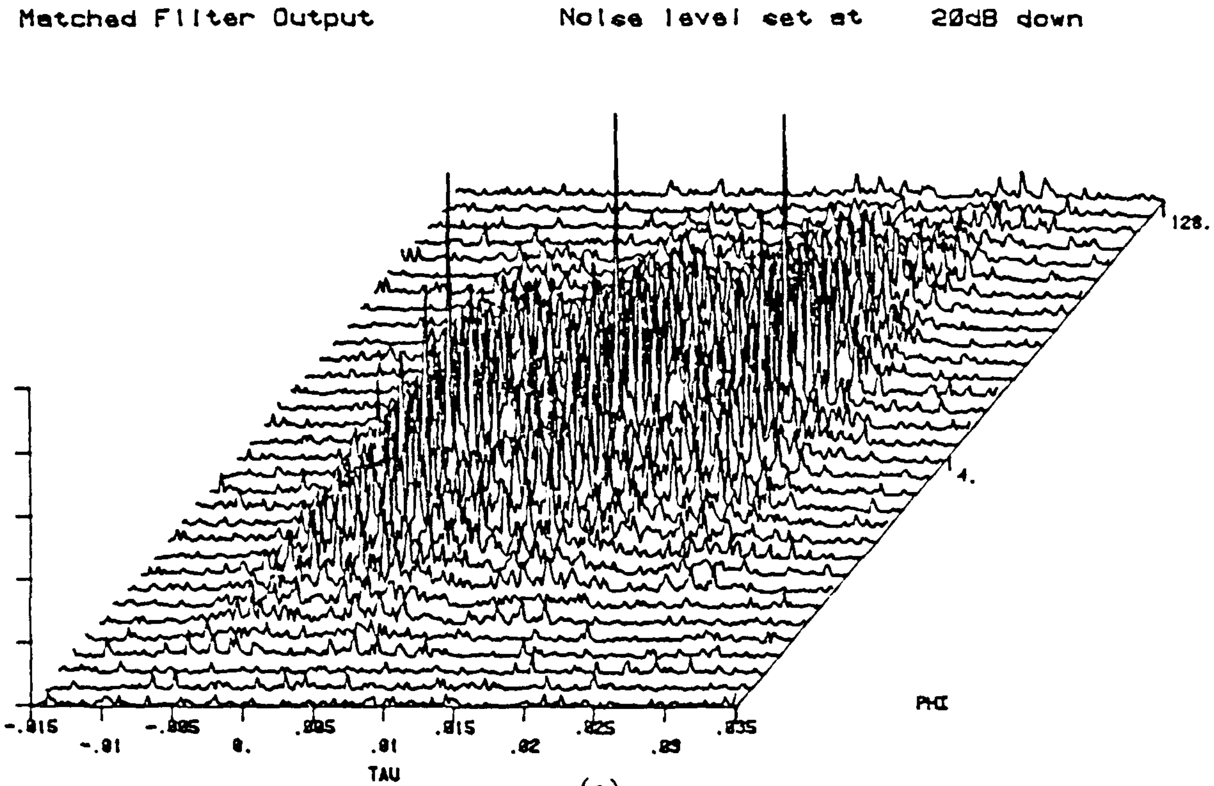
The choice for ε now becomes more critical. ST3 and SF3 were used and an SNR = 20dB chosen to form the next example. If ε is chosen too high, insufficient information remains to deconvolve, as shown earlier. There is now, however, a lower limit. If ε is set too low, the noise is amplified and the algorithm produces poor results as is clearly seen in Figure (4-6). It would be helpful to have a general idea of where to set the threshold, given an SNR.

Toward this end, one important property of the auto-ambiguity function is fundamental. This is the self-transform property[2]. Mathematically,

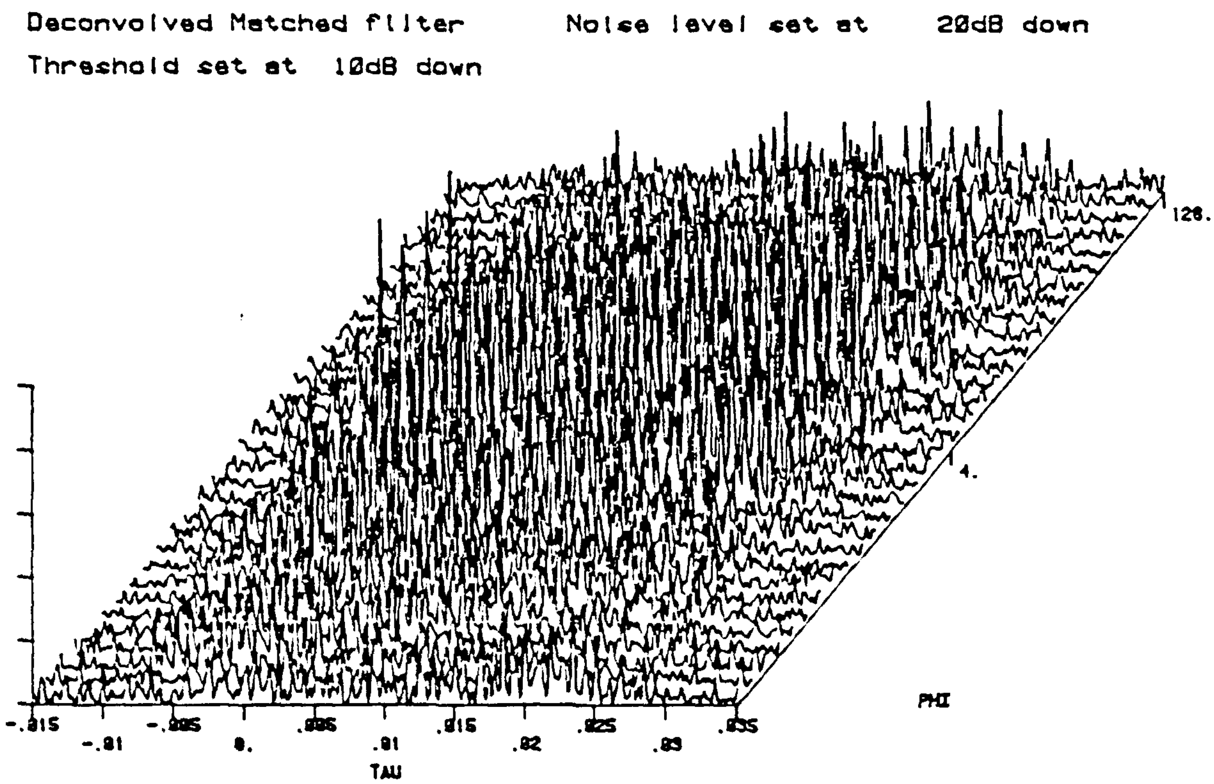
$$\int_{-\infty}^{\infty}\int a(\tau,\phi)e^{-2\pi j[\phi\nu-\tau\sigma]}d\tau d\phi = a(\nu,\sigma). \qquad (4\text{-}57)$$

This is important, because now the singular value spread, that is, the ratio of the largest to smallest 2DFT coefficients of the ambiguity function, is equal to the ratio of the largest to smallest ambiguity function values. The SNR can, therefore, be defined with respect to either the ambiguity function, or its 2DFT. Finally, since the matched filter is the result of a linear operation (the convolution), the threshold may also be related to the peak of the matched filter. It, therefore, makes sense to set the threshold somewhere around the average noise level defined as the SNR.

In this manner, if the matched filter output SNR is, for instance, 20dB, then the threshold should be set relative to this 20dB down level. Figure (4-6) shows the algorithm's performance on a matched filter with



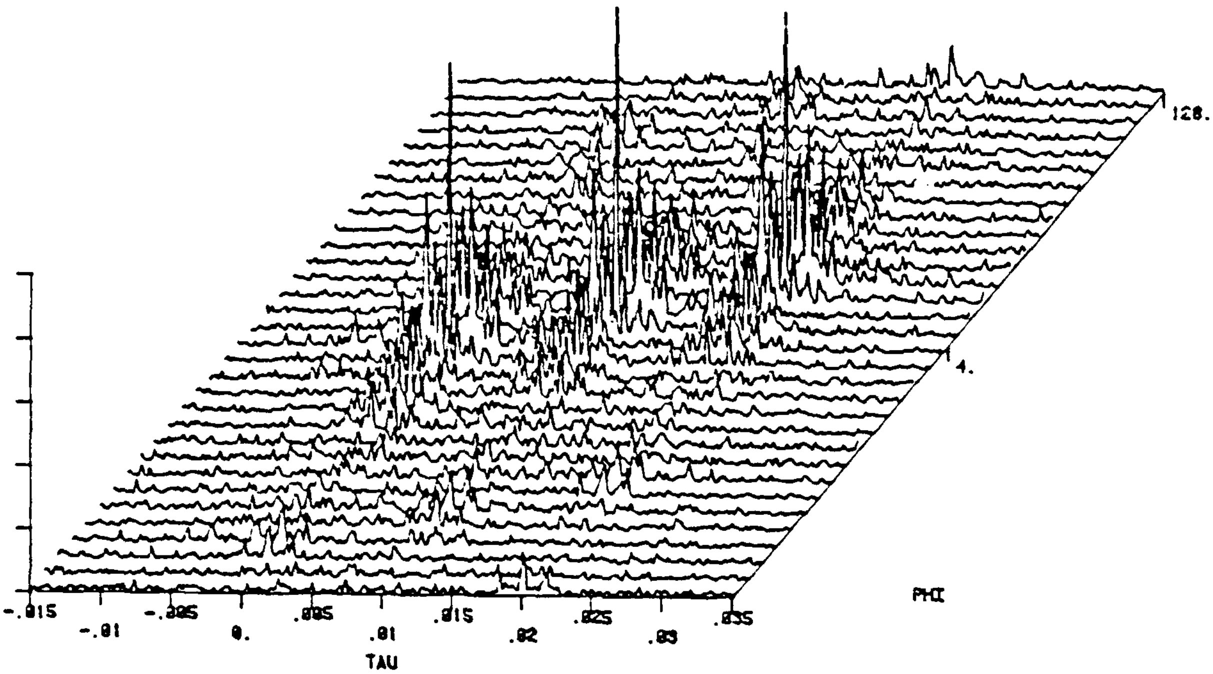
(a)



(b)

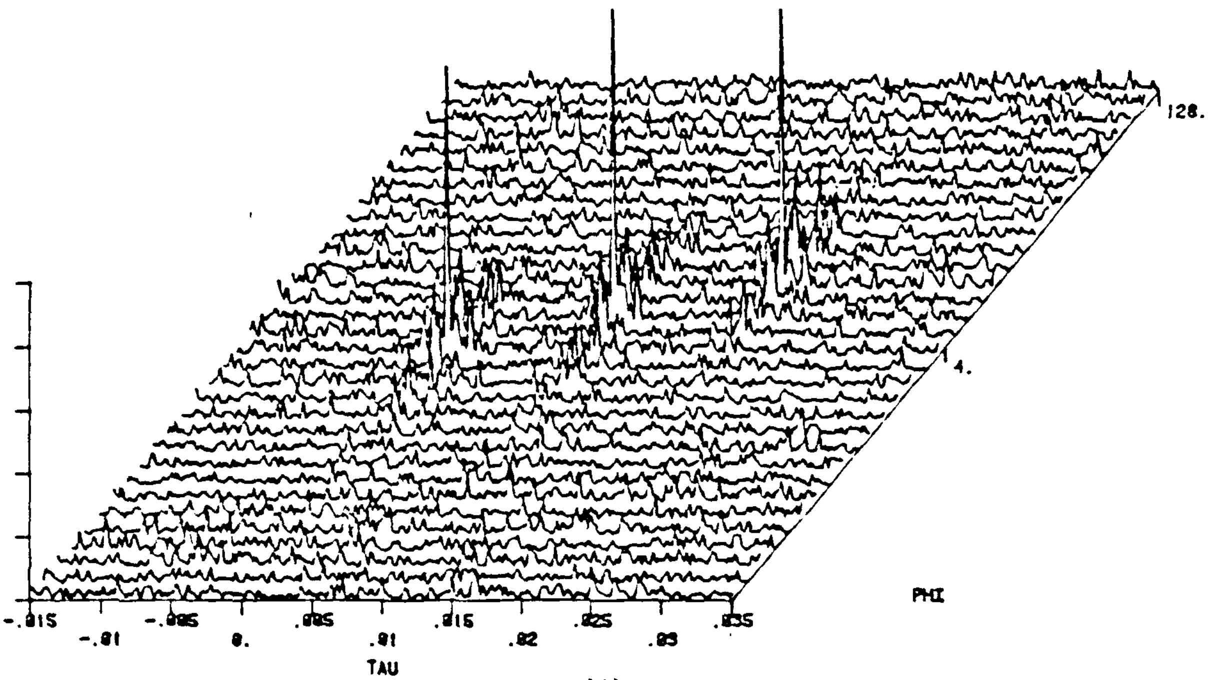
Figure 4-6 Example 4: Performance of the algorithm with noise in the matched filter.

Deconvolved Matched filter   Noise level set at   20dB down
Threshold set at 20dB down



(c)

Deconvolved Matched filter   Noise level set at   20dB down
Threshold set at 26dB down

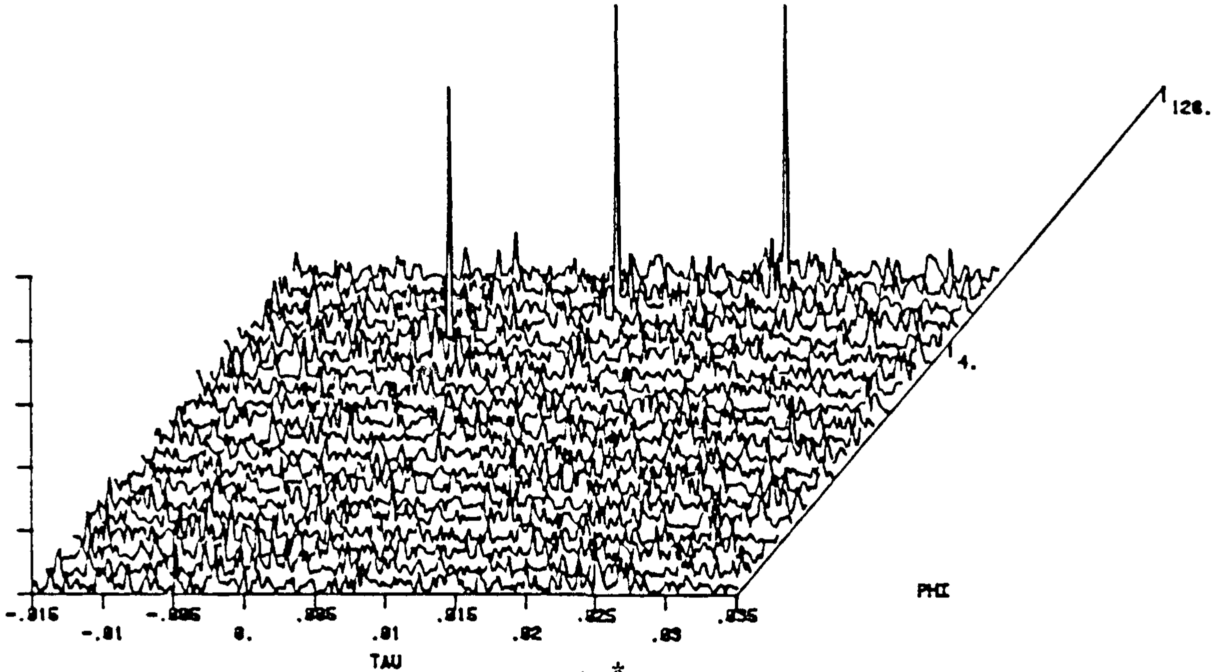


(d)

Figure 4-6 Continued

Deconvolved Matched Filter Noise level set at 20dB down

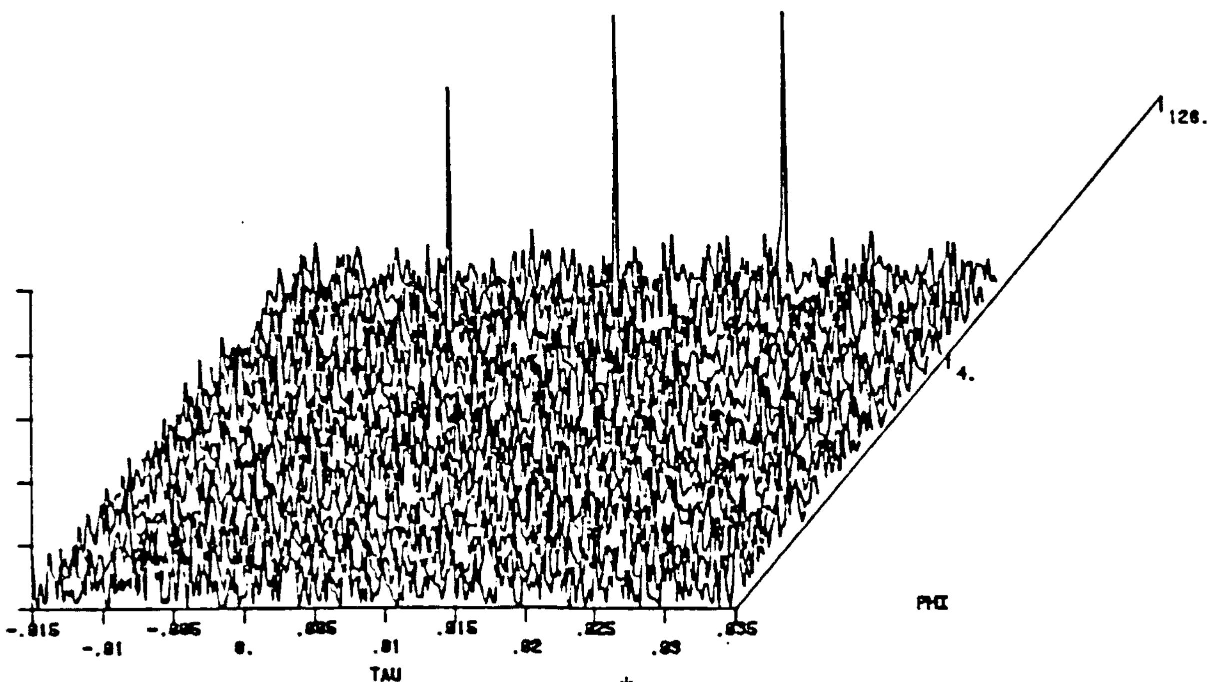
Threshold set at 30dB down



(e)*

Deconvolved Matched Filter Noise level set at 20dB down

Threshold set at 35dB down



(f)*

Figure 4-6 Continued

*For clarity, the slices behind the last highlight have been omitted.

a 20dB SNR. It should be noted that since the CAW signal has an effective processing gain of about 15.5dB, a 20dB output SNR represents a 4.5dB input SNR which represents a relatively noisy return.

Figure (4-6a) shows the matched filter output. The threshold is slowly decreased from -10dB to -40dB. At -10dB, the algorithm produces a poor estimate as expected. Through -20dB, the algorithm effectively cleans up the image. On the average, the noise has no effect up to this point. As ε is decreased more, the noise starts appearing in the estimate until at -40dB, the scattering function is almost completely hidden by the noise.

The expected error in the estimate can be found by considering the parameterized equation

$$(\underline{A} + \delta\underline{F})\underline{S}(\delta) = (\underline{M} + \delta\underline{f}) \qquad (4\text{-}58)$$

where

$$\underline{S}(0) = \underline{S}. \qquad (4\text{-}59)$$

The noise added to the matched filter is contained in the vector $\underline{f}$ and the effect of truncating the SVD expansion is quantified by the matrix $\underline{F}$. Employing a Taylor expansion on $\underline{S}(\delta)$ about zero,

$$\underline{S}(\delta) = \underline{S} + \delta\dot{\underline{S}}(0) + 0(\delta^2). \qquad (4\text{-}60)$$

Making a few substitutions it is easily shown that[6]

$$\frac{\|S(\delta) - S\|}{\|\underline{S}\|} < \kappa\,(\underline{A})\ \left[\delta\frac{\|F\|}{\|\underline{A}\|} + \delta\frac{\|f\|}{\|\underline{M}\|}\right], \qquad (4\text{-}61)$$

where

$$\kappa(\underline{A}) = \frac{\sigma_{max}}{\sigma_{min}}, \qquad (4\text{-}62)$$

the conditioning number of the matrix $\underline{A}$. This places an upper bound on the error in the estimate. It was assumed that the $O(\delta^2)$ terms are negligible.

Returning to the definition of the SVD,

$$\underline{A} = \sum_{i=1}^{N} \sigma_i \, \overline{u}_i \, \overline{u}_i^T \tag{4-63}$$

where the $\overline{U}_i$'s are the singular vectors assuming $\underline{A}$ is a real symmetric matrix. The sum can be split into two sums,

$$\underline{A} = \sum_{\sigma_i > \varepsilon} \sigma_i \, \overline{u}_i \, \overline{u}_i^T + \sum_{\sigma_i < \varepsilon} \sigma_i \, \overline{u}_i \, \overline{u}_i^T \, , \tag{4-64}$$

the right term containing the singular values which were taken out of the SVD expansion due to truncation. Therefore,

$$\underline{F} = -\sum_{\sigma_i < \varepsilon} \sigma_i \, \overline{u}_i \, \overline{u}_i^T \tag{4-64}$$

and since truncation also reduces the conditioning number,

$$\kappa(\underline{A}) = \frac{\sigma_{max}}{\sigma_{min}}, \qquad \sigma_{min} > \varepsilon \tag{4-65}$$

the smallest singular value now being equal to or greater than the threshold value ε. Equation (4-61) is now,

$$\frac{\| \underline{S}(\delta) - \underline{S} \|}{\| \underline{S} \|} < \frac{\sigma_{max}}{\sigma_{min}} \left[\delta \, \frac{\left\| \sum_{\sigma_i < \varepsilon} c_i \, \overline{u}_i \, \overline{u}_i \right\|}{\| \underline{A} \|} + \delta \, \frac{\| \underline{f} \|}{\| \underline{M} \|} \right] . \tag{4-66}$$

To support the validity of this result, there are three limiting cases to consider. The first is the case of no noise in the matched filter. Since $\|\underline{f}\| = 0$, the threshold ε can be set to zero at which point the term $\|\underline{F}\| = 0$ and as long as $\sigma_{min} > 0$, the error bound is zero.

Now assume there is noise, i.e., $\|\underline{f}\| \neq 0$. As ε approaches its maximum value of σ_{max}, the term $\|\underline{F}\|$ gets large and hence the error bound grows large. This was observed in the preceding examples for a threshold set too high.

Lastly, let ε go to zero and assume, as was the case in the examples, that $\underline{A}$ is ill-conditioned, that is $\kappa(\underline{A})$ is large. The term $\|\underline{F}\|$ goes to zero but as σ_{min} gets smaller, $\sigma_{max}/\sigma_{min}$ grows large and in multiplying by $\|\underline{f}\|$, the error bound again grows large. This was observed as the result of setting the threshold too low.

Thus it has been shown that Equation (4-66) is in agreement with the actual results demonstrated in the examples. Considering this equation term by term, the general trends can be seen. Consider the error versus threshold (ε) behavior of the conditioning number multiplied by the noise ($\|f\|$) term. For a given matched filter, the noise term is constant. If the threshold is set to zero, this term is at a maximum. As ε is increased, the conditioning number decreases monotonically to a value of one where this term reaches its minimum.

Next, consider the conditioning number multiplied by the truncation error ($\|F\|$) term. If the threshold is set to zero, $\|F\| = 0$ and so this term is at a minimum. As ε is increased, the error term increases and the conditioning number decreases to one. It is difficult to predict the actual behavior of this term but it is clear from the examples that this term has its minimum at $\varepsilon = 0$ and its maximum when the conditioning number reaches a value of one.

Summing the two terms, it becomes clear that there is a minimum error somewhere between the two extrema. This idea opens up a fascinating new topic which deserves further attention and it is the recommendation of the author that this be investigated in future work.

Chapter 5

Summary and Conclusions

The expected matched filter output from a communications channel characterized by a scattering function is considered. Treating this matched filter as an image, blurring due to the properties of the interrogating signal is modelled as a convolution process. Treating convolution as a multiple band-pass filtering operation, the process of deconvolution becomes the problem of finding the inverse filter. This is the basis for the standard spectral division method of deconvolution. The spectrum of the filter modelling the convolution process is simply inverted to yield the inverse filter. Problems arise when the original filter has a wide dynamic range as division by small numbers results in amplification of any noise in those spectral regions.

The thesis details a formulation of the problem employing an SVD to produce the inverse filter. Written as a simple matrix multiplication, the deconvolution is performed by finding a pseudo-inverse matrix used to remove the effects of the ambiguity function. The problem was first formulated to take advantage of the highly tractable nature of the BCCB matrix form. It was shown that deconvolution via the pseudo-inverse method is identical to deconvolution via spectral division.

If the matrix to be inverted is ill-conditioned, the pseudo-inverse deconvolution becomes an inherently noisy process. The reason is due to a wide dynamic range in the ambiguity function spectrum, identical to the cause of problems in the spectral division method. The singular values are shown to be equal to the values of the spectral components of the ambiguity function. Spectral regions with small values have a low SNR in a noisy matched filter; hence, the inverse filter accentuates the noise. Attempting to diminish the problem, the SVD expansion is truncated.

As a result of the truncation, resolution is lost in the deconvolution process. There is, therefore, a trade-off between minimization of the noise and retained resolution of the process output. This is shown graphically. A short analysis of the expected error provides an equation which was shown to properly predict the trends. More importantly, the equation suggests the existence of a threshold value which minimizes the error. It is highly suggested that this topic be given attention in furthering this thesis. Such a relationship can prove to be very useful when applying a deconvolution algorithm.

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